Exploration Efficiency of Random Walks in Dynamic Graph

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***Abstract*—Random walks on graphs have wide-ranging applications, including target node search on graphs and structural exploration of graphs. In recent years, analyses have been conducted not only on random walks over static graphs (where the topology does not evolve during the random walk) but also on dynamic graphs (where the topology evolves during the random walk). It has been shown that the behavior of a random walk on a random evolving graph is similar to that of a random walk on a static graph in that the stationary distribution remains unchanged over time. However, the impact of the dynamics (rate of change) of a dynamic graph on the characteristics of random walks, particularly the hitting time, has not been fully investigated. In this paper, we aim to quantitatively elucidate the search and exploration efficiency of random walks on various dynamic graphs. Specifically, we analyze the properties of random walks on a random evolving graph, and examine the impact of graph dynamics on search and exploration efficiency. As a consequence, we show that as the rate of change of the graph increases, the hitting time of the random walk operating on it decreases rapidly.**

***Index Terms*—Random Walk, Hitting Time, Cover Time, Evolving Graph, Dynamic Graph**

I. Introduction

A random walk on a graph has wide-ranging applications, including target node search and structural exploration of graphs. In recent years, analyses have been conducted not only on random walks over static graphs (where the topology does not change during the random walk) but also on dynamic graphs (where the topology changes during the random walk) [1-3]. As many real-world networks evolve over time, there has been growing interest in random walks on dynamic graphs.

It has been shown that the behavior of random walks on a random evolving graph, in which the stationary distribution does not change over time, is similar to that on a static graph. Additionally, analyses of the properties of random walks on dynamic graphs have been conducted. For instance, it has been shown that the mixing time and hitting time on a dynamic random regular graph are $O(n^2)$ [4]. However, the impact of the dynamics (rate of change) of a dynamic graph on the characteristics of random walks, particularly the hitting time, has not been thoroughly investigated. Furthermore, it is necessary to evaluate, from a quantitative perspective, how much the search and exploration efficiency of random walks in dynamic graphs improves compared to static graphs.

This study aims to answer the following research questions.

- How does the rate of change in dynamic graphs affect the hitting time of random walks?
- How does the search and exploration efficiency of random walks vary depending on the type of dynamic graph?
- To what extent can the hitting time and cover time of random walks in dynamic random evolving graphs be approximated accurately using random regular graphs?
- How does the search and exploration efficiency of random walks differ between dynamic and static graphs as the graph size and density vary?

This study aims to quantitatively elucidate the search and exploration efficiency of random walks on various dynamic graphs. Specifically, we analytically and through simulations investigate the properties of random walks on the representative dynamic graph, namely, the random evolving graph, and examine the impact of graph dynamics on search and exploration efficiency.

The main contributions of this paper are summarized as follows.

- The impact of the rate of change on the hitting time of random walks in dynamic graphs has been quantitatively clarified.
- The hitting time of random walks in dynamic random evolving graphs has been approximated using random regular graphs.

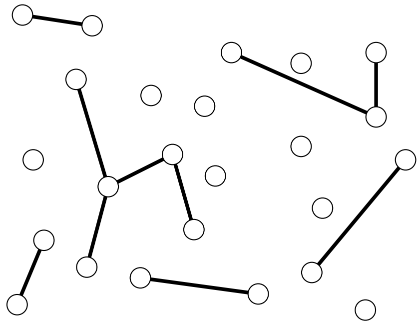
The structure of this paper is as follows. Section II introduces random walks on dynamic graphs. In Section III, we first clarify the differences in the characteristics of random walks on dynamic and static graphs through simulations, and then clarify the impact of graph dynamics on the characteristics of random walk. Section IV analytically derives the mean hitting time based on the rate of change of the graph. In Section V, the validity of our approximate analysis is examined. Finally, Section VI summarizes this paper and discusses the conclusions.

II. Random Walks on a Dynamic Graphs

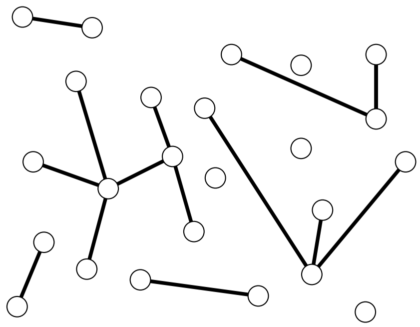
In this study, the dynamic graph under consideration is an unweighted, undirected graph $G_n = (V_n, E_n)$ whose topology changes at each slot $n \geq 1$. Here, the set of nodes remains constant (i.e., $V_1 = V_2 = V_3 = \dots$).

The graph G_n represents the graph at a specific time n. An example of a dynamic graph is shown in Fig. 1. G_n has the vertex set V_n and the edge set E_n. Furthermore, the condition $V_0 = V_1 = V_2 \dots$ holds, indicating that the vertex sets are identical at different times.

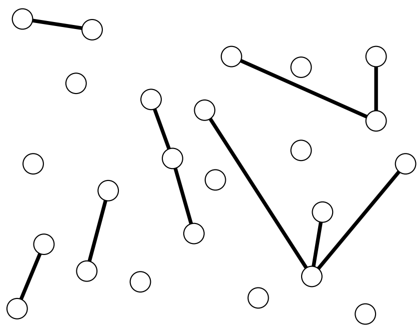
In a dynamic graph, the cover time of a discrete-time random walk is the time it takes for an agent, starting from an initial node on the graph G_n at slots $n = 0$, to transition over



(a) $G_1 = (V_1, E_1)$

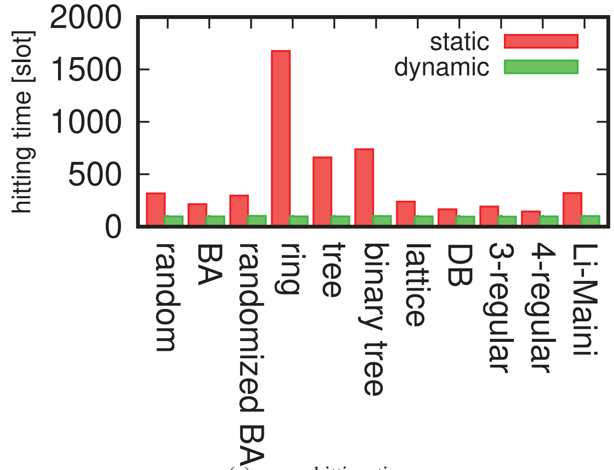


(b) $G_2 = (V_2, E_2)$

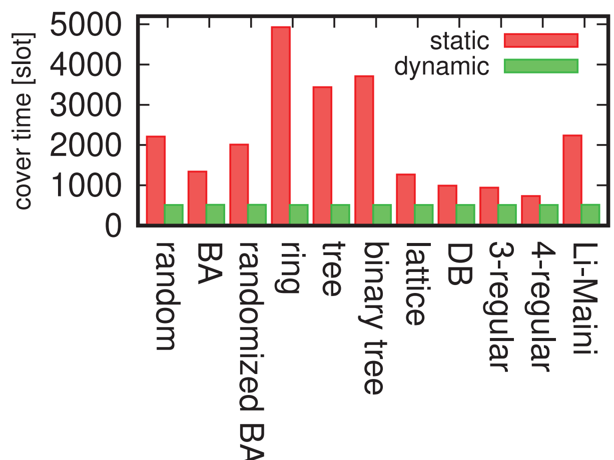


(c) $G_3 = (V_3, E_3)$

Fig. 1. Example of a dynamic graph (rate $\lambda = 1$, slots $n = 1, 2, 3$, node set $V_1 = V_2 = V_3$)



(a) mean hitting time



(b) mean cover time

Fig. 2. Differences in the characteristics of random walks on dynamic and static graphs $n = 100$

the graph G_n at slots $n > 0$ and visit all other nodes at least once, excluding the starting node.

In a dynamic graph, the hitting time of a discrete-time random walk is the time it takes for an agent, starting from an initial node on the graph G_n at slot $n = 1$, to transition over the graph G_n at slot n and reach the target node for the first time.

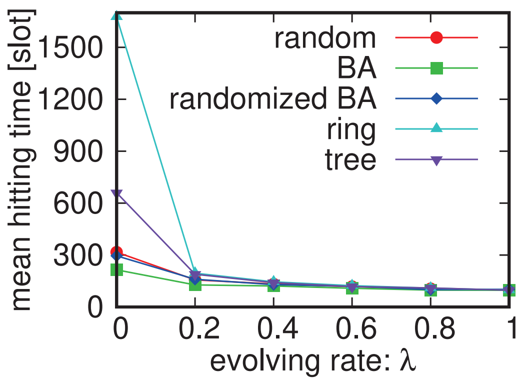
III. Experiment

A. Characteristics of Random Walks on Static and Dynamic Graphs

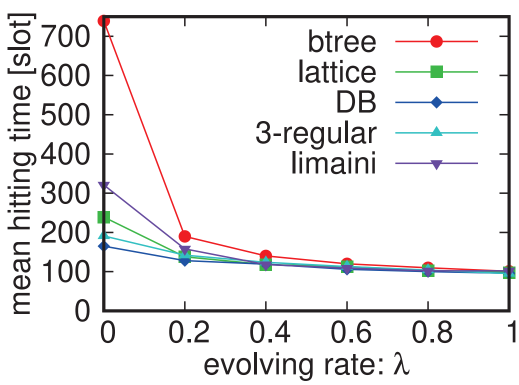
In the following, we analyze the impact of graph dynamics on the exploration efficiency of a simple random walk (SRW) across graphs with different topological structures.

We measured the average hitting time and cover time of a simple random walk on both static and dynamic graphs generated using several network generation models. To generate the graphs, we used 10 different network generation models, producing random regular graphs, BA (Barabási–Albert) graphs, randomized BA graphs, ring graphs, tree graphs, binary tree graphs, lattice graphs, 3-regular graphs, 4-regular graphs, DB (Degree-Bounded) graphs, and Li-Maini graphs. Using each network generation model, we generated 1,000 graphs. One of these graphs was designated as the static graph G, and a dynamic graph G_n was created by randomly switching among the 1,000 graphs at each step. For both a given static graph and a dynamic graph, we measured the hitting time and cover time of a simple random walk, where an agent initiated movement from a randomly selected starting node s. In both the static and dynamic graphs, we performed 10,000 trials of the simple random walk starting from node s to calculate the average and the 95% confidence intervals of the hitting time and the cover time.

Fig. 2 shows two types of exploration efficiency (average hitting time and cover time) of a simple random walks on 10 types of static graphs and their corresponding dynamic graphs. From these results, it is evident that, for all graph types, the exploration efficiency of a simple random walk on dynamic graphs is significantly better than that on static graphs. It is also observed that the average hitting time on dynamic graphs approximately equals the number of nodes, which is 100.

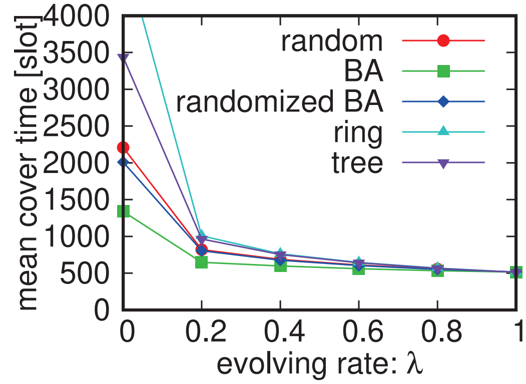


(a)

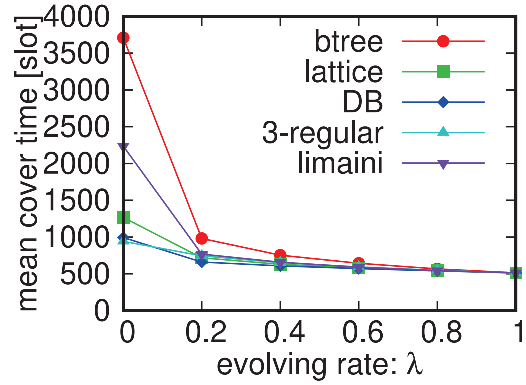


(b)

Fig. 3. Relationship between the Rate of λ and Hitting Time



(a)



(b)

Fig. 4. Relationship between the Rate of λ and Cover Time

B. Impact of Graph Dynamics on the Characteristics of Random Walks

We analyze the impact of the rate of change λ in random evolving graphs on the hitting time and cover time of random walks across graphs with different topological structures. To measure the average hitting time and cover time of a simple random walks on static and dynamic graphs, we generated graphs with 100 nodes using several network generation models.

Specifically, we generated 10 types of graphs: random regular graphs, BA (Barab'asi–Albert) graphs, randomized BA graphs, ring graphs, tree graphs, binary tree graphs, lattice graphs, 3-regular graphs, 4-regular graphs, DB (Degree-Bounded) graphs, and Li-Maini graphs. For each model, 1,000 graphs were generated, and at each step, the graph was switched with a probability determined by the rate of change λ.

On the 10 types of generated graphs, the rate of change λ was set to values ranging from 0 to 1, and 10,000 trials of a simple random walks were conducted starting from each node. Through this experiment, the average hitting time and cover time, along with the 95% confidence intervals, were determined.

The results for the hitting time and cover time of a simple random walk on evolving graphs generated with 10 types of network generation models are shown in Fig. 3 and 4, respectively. From these results, it was confirmed that as the rate of change λ of the graph increases, the hitting time and cover time of the random walk decrease rapidly. It was found that, regardless of the network generation model of the evolving graph, the hitting time converges to nearly the same value across all network generation models when the rate of change $\lambda = 1$ (i.e., the state in which the graph undergoes the most rapid changes). Similarly, the cover time was also shown to exhibit nearly identical values irrespective of the network generation model. Furthermore, it was revealed that the hitting time under these conditions is approximately equal to the number of nodes in the graph.

IV. Analysis

Let the rate of change of a random evolving graph G_n be denoted by λ, its average degree by k, and the n-th graph in G by G_n for $n \geq 1$. Since G is a random evolving graph, all G_n share the same probability distribution, and G_n and G_{n+1} are mutually independent.

When the rate of change $\lambda \gg 0$, each G_n in the random evolving graph is independent of one another, allowing an agent traversing the random evolving graph to perceive the graph structure as a random regular graph with a certain degree.

Here, let the effective average degree observed by an agent over L slots be $\bar{k}$. Given that the average degree of the graph G_n is k, the effective average degree when the graph does not change over L slots is k. Similarly, if the graph changes i times over L slots, the effective average degree is $k\,(i+1)$.

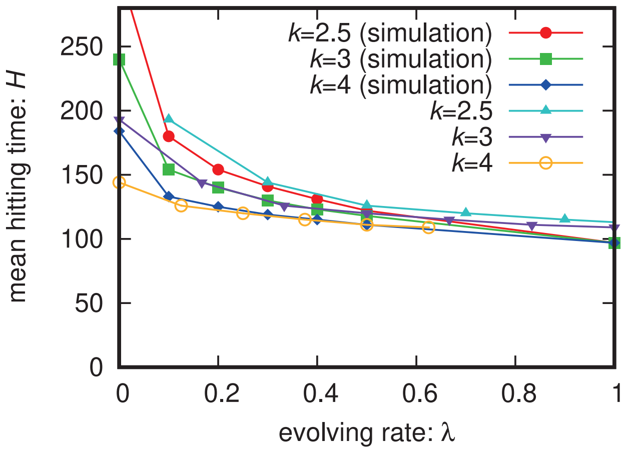


Fig. 5. Relationship between the rate of change λ in a random evolving graph and the average hitting time.

The probability p_i that the graph changes i times over L slots is

$$p_i = \binom{L}{i}(1-\lambda)^{L-i}\lambda^i. \quad (1)$$

Thus, the apparent degree $\overline{k}$ observed by the agent is

$$\overline{k} = \sum_{i=0}^{L} k(1+i)p_i. \quad (2)$$

Expanding this equation, we have

$$\overline{k} = k\sum_{i=0}^{L}(1+i)\binom{L}{i}(1-\lambda)^{L-i}\lambda^i. \quad (3)$$

Here, the degree when the graph does not change over L slots is k, and when it changes, the degree is $k(1+i)$. Therefore, the average degree can be approximated as follows.

$$\overline{k} \approx k\left(\frac{1}{L}+\lambda\right) \quad (4)$$

If we set the observation period for the apparent degree $\overline{k}$ to $L = 1/\lambda$, then $\overline{k} = 2\,k\,\lambda$.

$$\overline{k} = 2k\lambda \quad (5)$$

Therefore, if we denote the expected hitting time in a random regular graph with degree d as $H(d)$, the expected hitting time H in the random evolving graph G is approximately given by

$$H \simeq H(\overline{k}) = H(2\,k\,\lambda) \quad (6)$$

V. Numerical Examples

Through several numerical examples, we analyze the impact of the rate of change λ in a random evolving graph on the hitting time of a random walk.

We calculate the expected hitting time based on Eq. (6) for varying rates of change λ in a random evolving graph G, where each G_n constituting G is a random network with 100 nodes generated by the ER model (Fig. 5). The figure shows results when the average degree of the random evolving graph G (i.e., the average degree of each graph G_n comprising it) is varied to 2.5, 3, and 4. To verify the validity of the analytical results, the average hitting times measured through simulation experiments under the same conditions are also presented. In the simulation, a random network with 100 nodes was generated using the ER model. The hitting time was measured for a simple random walk initiated from a randomly selected starting node within the random network. Additionally, the average hitting time for a simple random walk was obtained by performing 1,000 trials from the starting node.

These results show that as the rate of change λ of the graph increases, the hitting time of the random walk operating on it decreases rapidly. Additionally, as the rate of change becomes larger, the hitting time asymptotically approaches the number of nodes in the graph. It is also confirmed that the analytical results generally agree with the simulation results.

VI. Conclusions

In this study, we had analyzed how the rate of change in dynamic graphs affects the characteristics of random walks, particularly the hitting time. In the experiments, we had measured the hitting time and cover time of random walks on both static and dynamic evolving graphs generated using 10 different network generation models with various topologies. The results show that, across the network generation models, the exploration efficiency of random walks on dynamic graphs is higher than that on static graphs, with the average hitting time on dynamic graphs tending to be nearly equal to the number of nodes. Additionally, it is confirmed that as the rate of change λ increases, both the hitting time and cover time decrease rapidly. Specifically, when the rate of change reaches its maximum, the hitting time and cover time converge to similar values, with the hitting time becoming approximately equal to the number of nodes.

Furthermore, it has been observed that analytical predictions of the hitting time for random walks on dynamic graphs with varying rates of change λ are consistent with the simulation results. This indicates a close relationship between the characteristics of random walks on dynamic graphs and the rate of change of the graph.

Acknowledgements

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