Performance Analysis of Recurrence Entanglement Distillation Protocol Under Asymmetric Conditions

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비대칭 환경에서의 리커런스 얽힘 정제 프로토콜 성능 분석

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Abstract

Standard analyses of the recurrence entanglement purification protocol typically assume symmetric input states, neglecting practical channel heterogeneity. This study investigates the protocol's performance under asymmetric input conditions. We derived generalized formulas for output fidelity and validated them through exact density matrix simulations using Qiskit. Our results demonstrate that fidelity mismatches between input pairs significantly reduce purification gain, underscoring the critical need for channel balancing strategies in realistic quantum networks.

Ⅰ. Introduction

Quantum entanglement is essential for scalable quantum networks [1], yet environmental noise necessitates entanglement distillation protocols (EDPs) to restore fidelity [2]. While the seminal recurrence protocol [3] and its variants [4] are widely adopted, standard analyses predominantly assume symmetric input states. However, practical implementations inevitably face channel heterogeneity [5], resulting in entangled pairs with asymmetric qualities that are often overlooked in theoretical models [6]. In this letter, we address this gap by deriving analytical formulas for the recurrence protocol under asymmetric conditions. Validated by exact Qiskit simulations, our results quantify the significant performance degradation caused by fidelity mismatches, highlighting the critical importance of channel balancing in quantum repeater designs.

Ⅱ. Recurrence Entanglement Distillation Protocol Under Asymmetric Conditions

We consider the standard recurrence entanglement distillation protocol where two parties, Alice and Bob, share two noisy entangled pairs. We assume the initial states are Werner states, characterized solely by their fidelities F. In a standard analysis, the two input pairs are assumed to have identical fidelity $F_1 = F_2$. However, to model realistic channel heterogeneity, we generalize this to an asymmetric condition where the two input pairs possess distinct fidelities, F_1 and F_2.

In the protocol, a bilateral CNOT gate is applied using the first pair as the control and the second pair as the target. Subsequently, the target pair is measured in the computational basis (Z-basis). If the measurement results coincide (both 0 or both 1), the control pair is kept; otherwise, it is discarded.

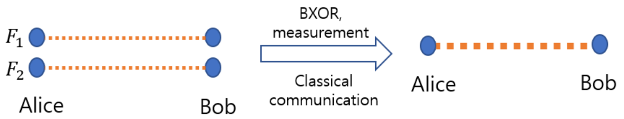

Fig.1. Schematic of the recurrence entanglement purification protocol applied to asymmetric inputs

Under symmetric conditions $F_1 = F_2 = F$, the output fidelity F_{out} and success probability P_{succ} are governed by the standard recurrence formulas [3]. We extend these to the asymmetric case. The probability of a successful distillation round, $P_{succ}(F_1, F_2)$, is derived as:

$$P_{succ} = F_1F_2 + \frac{1}{3}[F_1(1-F_2) + F_2(1-F_1)] + \frac{5}{9}(1-F_1)(1-F_2).$$

Correspondingly, the unnormalized output fidelity is given by the term $F_1F_2 + \frac{1}{9}(1-F_1)(1-F_2)$. The final output fidelity $F_{out}(F_1, F_2)$, is derived as:

$$F_{out} = \frac{F_1F_2 + \frac{1}{9}(1-F_1)(1-F_2)}{P_{succ}}.$$

It is straightforward to verify that when $F_1 = F_2 = F$, these equations reduce to the standard symmetric forms.

To validate the analytical derivation, we performed exact numerical simulations using the Qiskit DensityMatrix framework, which eliminates finite-sampling noise. The simulation modeled a four-qubit system representing two entangled pairs initialized as Werner states with fidelities F_1 and F_2. The purification circuit employed bilateral CNOT gates followed by parity measurements. Upon a successful parity check (coincident outcomes), the target qubits were traced out, and the fidelity of the remaining control pair was computed against the ideal Bell state. We evaluated the protocol under three distinct input conditions by sweeping the average input fidelity F_{avg} from 0.5 to 1.0. These scenarios included a symmetric control group ($F_1 = F_2$) and two asymmetric cases with fidelity deviations of $\Delta = 0.2(F_{avg} \pm 0.1)$ and $\Delta = 0.4(F_{avg} \pm 0.2)$, allowing for a direct assessment of how channel heterogeneity impacts purification performance.

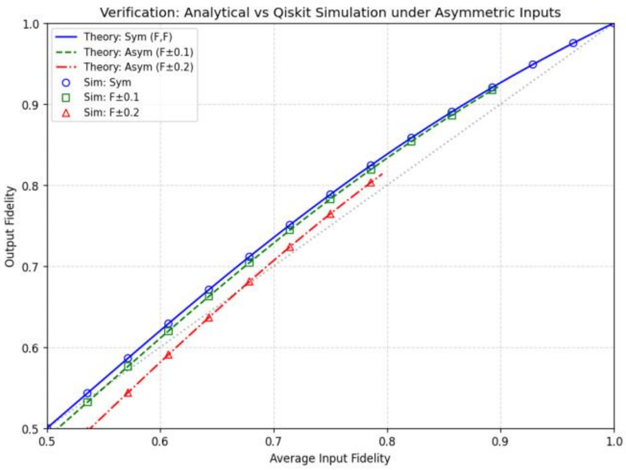

Fig.2. Comparison of analytical predictions (lines) and Qiskit simulation results (markers)

Fig. 2 presents the comparative results between the analytical predictions (lines) and the Qiskit simulation data (markers). The perfect alignment between the two confirms the validity of our derived asymmetric recurrence formulas. The results clearly indicate that for a fixed average input fidelity F_{avg}, the purification performance degrades as the asymmetry Δ increases. The symmetric case ($\Delta = 0$, blue line) yields the maximal output fidelity. In contrast, the highly asymmetric scenario ($\Delta = 0.4$, red line) exhibits a significant reduction in purification gain, particularly in the critical regime where $F_{avg} \in [0.6, 0.8]$. This suggests that mixing a high-quality pair with a low-quality pair introduces excessive noise during the bilateral CNOT operations, thereby lowering the probability of successful error detection and reducing the overall efficiency of the protocol.

Ⅲ. Conclusion

In this work, we investigated the impact of input state asymmetry on the performance of the recurrence entanglement purification protocol. We generalized the standard recurrence equations to account for unequal input fidelities and validated the theoretical model through exact density matrix simulations. Our findings demonstrate that fidelity mismatches between entangled pairs inevitably reduce the distilled output fidelity compared to the symmetric ideal. This highlights that in realistic quantum networks, where channel heterogeneity is common, maintaining balanced channel conditions or developing adaptive protocols is crucial for optimizing entanglement distribution.

ACKNOWLEDGMENT

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