Fidelity Behavior in GHZ State Distillation with Depolarizing Errors

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Depolarizing 오류 하에서 GHZ 상태 정제의 충실도 연구

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Abstract

This paper investigates the effectiveness of a distillation protocol for 3-qubit GHZ states under depolarizing noise. We analytically compute the output fidelity of the distilled state and verify its purification effect by comparing it to the initial fidelity before distillation. The results demonstrate that the protocol can successfully enhance the entanglement quality of noisy GHZ states within a specific noise range. The entire distillation process is simulated using Qiskit, providing numerical support for the theoretical analysis.

Ⅰ. Introduction

Entanglement Distillation Protocols (EDPs) play a central role in quantum communication and distributed quantum computing, allowing noisy entangled states to be purified into high-fidelity resources through local operations and classical communication (LOCC)[1]. While early studies focused primarily on bipartite entanglement[2,3], such as Bell states, there is a growing need to understand and process multipartite entangled states—especially Greenberger-Horne-Zeilinger (GHZ) states[4]—which are fundamental for quantum secret sharing, networked quantum systems, and fault-tolerant architectures[5].

In this paper, we analyze the distillation performance of 3-qubit GHZ states under depolarizing noise. We compute the output fidelity of the protocol theoretically and verify its effectiveness through numerical simulations. The simulation is performed using the local AerSimulator backend provided by Qiskit, allowing for high-precision modeling of ideal and noisy operations in a controlled environment.

Ⅱ. GHZ state and Distillation Protocol

A. 3-qubit GHZ state

The Greenberger–Horne–Zeilinger (GHZ) state is a canonical example of multipartite entanglement, representing a maximally entangled state shared among three or more qubits. Unlike bipartite entangled states such as the Bell pair, GHZ states exhibit non-classical correlations that cannot be reduced to pairwise entanglement, making them essential in quantum networking, secret sharing, and nonlocality tests.

In this study, we focus on the 3-qubit GHZ state, which takes the form

$$|GHZ\rangle = \frac{|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C}{\sqrt{2}}.$$

This state is used as the target of our entanglement distillation protocol. We investigate how its fidelity can be preserved or enhanced when subjected to depolarizing noise and subsequently processed through local operations.

B. Entanglement Distillation Protocol

Entanglement Distillation Protocols (EDPs) were first introduced in the late 1990s to enhance the quality of entangled states degraded by noise during transmission or storage. Since then, various EDPs have been proposed and developed, primarily focusing on bipartite entanglement. In this work, we analyze a distillation protocol tailored for 3-qubit GHZ states.

As shown in Fig. 1(a), the protocol begins with Alice generating two ideal 3-qubit GHZ states. In each GHZ state, two of the qubits are sent through quantum channels to Bob and Charlie, respectively, while one qubit is retained by Alice. These channels are modeled as depolarizing noise channels with error probability p, where each qubit independently undergoes an X, Y, or Z error with equal probability. After transmission, the three parties share two noisy copies of the GHZ state, and the density operator representation ρ of the overall quantum state can be expressed as follows:

$$\rho' = (1-p)|GHZ'\rangle\langle GHZ'| + \frac{p}{3}(IXI \otimes |GHZ'\rangle\langle GHZ'| + IYI \otimes |GHZ'\rangle\langle GHZ'| + IZI \otimes |GHZ'\rangle\langle GHZ'|),$$

where

$$|GHZ'\rangle\langle GHZ'| = (1-p)|GHZ\rangle\langle GHZ| + \frac{p}{3}XII \otimes |GHZ\rangle\langle GHZ| + \frac{p}{3}YII \otimes |GHZ\rangle\langle GHZ| + \frac{p}{3}ZII \otimes |GHZ\rangle\langle GHZ|.$$

Next, as illustrated in Fig. 1(b), each party performs a CNOT gate between their two local qubits: the control qubit corresponds to one copy of the GHZ state (the "source"), and the target qubit corresponds to the other (the "target"). Following the CNOT operations, all parties measure their respective target qubits in the computational basis. The measurement outcomes are then shared through classical communication.

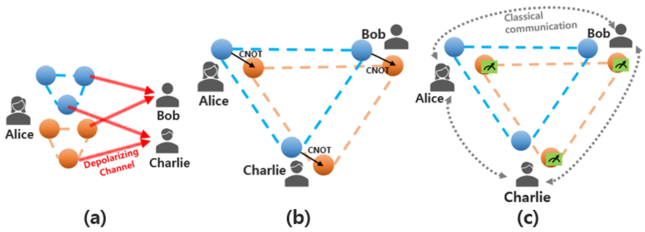

Fig. 1. Entanglement distillation of 3-qubit GHZ state

The protocol is considered successful if all three measurement results are equal—i.e., either "000" or "111". In this case, the remaining source GHZ state is kept as the distilled output. If the measurement outcomes do not match, the state is discarded and the protocol is considered a failure.

Ⅲ. Simulation and Results

To evaluate the performance of the GHZ distillation protocol under depolarizing noise, we numerically simulated the process using Qiskit's AerSimulator. We assumed that two copies of noisy 3-qubit GHZ states were distributed through depolarizing channels with varying noise strength p. Each party then applied CNOT gates and performed projective measurements as described in the protocol.

Fig. 2 shows the simulated output fidelity as a function of the input fidelity (determined by the depolarizing error rate). The simulation results agree well with the theoretical predictions and clearly show a significant fidelity improvement compared to the undistilled case. This confirms the effectiveness of the protocol in enhancing entanglement quality over a broad noise range

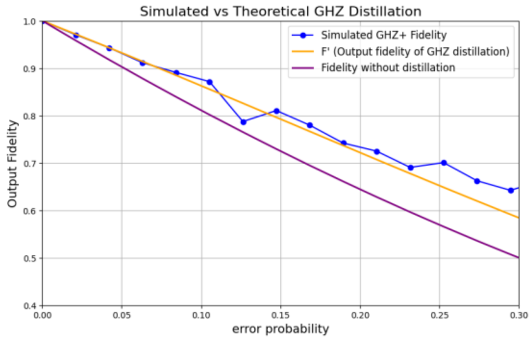

Fig. 2. Output fidelity of 3-qubit GHZ state after distillation compared with theoretical prediction and undistilled fidelity

Ⅳ. Conclusion

In this work, we analyzed a distillation protocol for 3-qubit GHZ states affected by depolarizing noise. Both theoretical and simulated results indicate that the protocol can improve the fidelity of noisy GHZ states. Our findings suggest that such multipartite entanglement distillation schemes are promising for quantum communication and computation tasks in realistic noisy environments.

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