Linear-Optical Realization of Repeated Helstrom Measurements for Quantum State Discrimination

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***Abstract*—We present a mathematical formulation for implementing the Helstrom measurement for quantum state discrimination using linear optical devices. It is known that performing collective measurements on n copies of a quantum state can reduce the minimum error probability. However, such collective measurements typically require nonlinear optical interactions. Here, we show theoretically that the Helstrom measurement can be implemented on a linear optical platform through a sequence of single-copy repeated Helstrom measurements.**

I. Introduction

In minimum-error quantum state discrimination [1], also known as the Helstrom measurement, we consider two non-orthogonal quantum states $\boldsymbol{\varXi}_0$ and $\boldsymbol{\varXi}_1$. Suppose Alice prepares one of these states and sends it to Bob, whose task is to determine which state was sent. To do so, Bob constructs the Helstrom operator

$$\boldsymbol{\Delta} = q\boldsymbol{\varXi}_0 - (1-q)\boldsymbol{\varXi}_1, \tag{1}$$

where q denotes the prior probability that $\boldsymbol{\varXi}_0$ was sent. The minimum achievable error probability is given by the Helstrom bound

$$P_{\min}^{(\mathrm{err})} = \frac{1}{2}\left[1 - \|\boldsymbol{\Delta}\|_1\right], \tag{2}$$

where $\|\boldsymbol{\Delta}\|_1 = \mathrm{tr}\left(\sqrt{\boldsymbol{\Delta}^\dagger\boldsymbol{\Delta}}\right)$ is the trace norm of $\boldsymbol{\Delta}$.

The optimal measurement associated with $\boldsymbol{\Delta}$ is described by a two-element positive operator-valued measure (POVM),

$$\boldsymbol{M}_0 = \sum_{\lambda_i \geq 0} |\phi_i\rangle\langle\phi_i|, \qquad \boldsymbol{M}_1 = \sum_{\lambda_i < 0} |\phi_i\rangle\langle\phi_i|, \tag{3}$$

where λ_i and $|\phi_i\rangle$ are the eigenvalues and eigenvectors of $\boldsymbol{\Delta}$, respectively. The POVM elements satisfy

$$\boldsymbol{M}_0 + \boldsymbol{M}_1 = \boldsymbol{I}, \tag{4}$$

where $\boldsymbol{I}$ is the identity operator.

A POVM can equivalently be written in terms of Kraus operators $\{\boldsymbol{K}_i\}$ as

$$\boldsymbol{M}_i = \boldsymbol{K}_i^\dagger\boldsymbol{K}_i, \qquad \sum_i \boldsymbol{K}_i^\dagger\boldsymbol{K}_i = \boldsymbol{I}. \tag{5}$$

When outcome i occurs, the post-measurement state is

$$\boldsymbol{\varXi}'_{j\to i} = \frac{\boldsymbol{K}_i\boldsymbol{\varXi}_j\boldsymbol{K}_i^\dagger}{\mathrm{tr}(\boldsymbol{K}_i\boldsymbol{\varXi}_j\boldsymbol{K}_i^\dagger)}. \tag{6}$$

The minimum error probability can be further reduced if multiple identical copies of the state are available. If Alice prepares $\boldsymbol{\varXi}_0^{\otimes n}$ or $\boldsymbol{\varXi}_1^{\otimes n}$, the corresponding Helstrom bound becomes

$$P_{\min}^{(\mathrm{err})}(n) = \frac{1}{2}\left[1 - \left\|q\boldsymbol{\varXi}_0^{\otimes n} - (1-q)\boldsymbol{\varXi}_1^{\otimes n}\right\|_1\right]. \tag{7}$$

In the asymptotic limit $n\to\infty$, the error probability obeys the quantum Chernoff bound [2]:

$$\lim_{n\to\infty} \frac{-\ln P_{\min}^{(\mathrm{err})}(n)}{n} = \zeta(\boldsymbol{\varXi}_0, \boldsymbol{\varXi}_1), \tag{8}$$

where the error exponent is

$$\zeta(\boldsymbol{\varXi}_0, \boldsymbol{\varXi}_1) = -\ln\left[\min_{0\leq z\leq 1} \mathrm{Tr}\left(\boldsymbol{\varXi}_0^z\boldsymbol{\varXi}_1^{1-z}\right)\right]. \tag{9}$$

Consequently,

$$P_{\min}^{(\mathrm{err})}(n) \sim e^{-n\zeta(\boldsymbol{\varXi}_0,\boldsymbol{\varXi}_1)}. \tag{10}$$

While collective measurements on n copies achieve the minimum error allowed by the quantum Chernoff bound, such measurements are experimentally demanding. It has been shown in [3] that near-optimal performance can be achieved by repeated partial collective measurements. This method operates by performing collective measurements on $m \ll n$ copies, repeated n/m times, with the final outcome determined by majority voting. The protocol asymptotically attains the same error exponent as the optimal collective measurement.

Here, we briefly demonstrate how the repeated Helstrom measurement can be implemented on one of the simplest quantum platforms, which is a linear optical device consisting only of beam splitters and phase shifters. We focus on the simplest case of $m = 1$, corresponding to single-copy repeated Helstrom measurements.

II. Implementing the Repeated Helstrom Measurements with Tunable Linear Optics

A. Decomposition of POVM Operators

Any square matrix $\boldsymbol{A}$ can be decomposed as $\boldsymbol{A} = \boldsymbol{V}^\dagger\boldsymbol{D}\boldsymbol{U}$, where $\boldsymbol{U}, \boldsymbol{V}$ are unitary and $\boldsymbol{D}$ is diagonal. Thus, a Kraus operator $\boldsymbol{K}_i$ can also be expressed as $\boldsymbol{K}_i = \boldsymbol{V}_i^\dagger\boldsymbol{D}_i\boldsymbol{U}_i$. Consequently,

$$\boldsymbol{M}_0 = \boldsymbol{K}_0^\dagger\boldsymbol{K}_0 = \boldsymbol{U}_0^\dagger\boldsymbol{D}_0^\dagger\boldsymbol{D}_0\boldsymbol{U}_0, \tag{11}$$

$$\boldsymbol{M}_1 = \boldsymbol{I} - \boldsymbol{M}_0 = \boldsymbol{U}_0^\dagger(\boldsymbol{I} - \boldsymbol{D}_0^\dagger\boldsymbol{D}_0)\boldsymbol{U}_0. \tag{12}$$

Consider a single-photon polarization state

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle, \tag{13}$$

where $|H\rangle$ and $|V\rangle$ denote horizontal and vertical polarization, respectively. As shown in [4], a photonic circuit consisting only of linear optical elements (beam splitters, polarization rotators, and phase shifters) can perform the following transformations:

$$\boldsymbol{D}_0 : \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \mapsto \begin{bmatrix} e^{i\epsilon_0}\cos\theta & 0 \\ 0 & \cos\phi \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \tag{14}$$

$$\boldsymbol{D}_1 : \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \mapsto \begin{bmatrix} e^{i\epsilon_1}\sin\theta & 0 \\ 0 & \sin\phi \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{15}$$

Here, ϵ_0, ϵ_1 are phase shifts and θ, ϕ are rotation angles. The detection probabilities at the two output ports G_0 and G_1 are given by

$$P_{G_0} = \mathrm{tr}(\boldsymbol{\Xi}\boldsymbol{M}_0), \qquad P_{G_1} = \mathrm{tr}(\boldsymbol{\Xi}\boldsymbol{M}_1).$$

B. Parameter Optimization for Linear Optics

The Helstrom bound can equivalently be expressed as an optimization over POVM elements:

$$P_{\min}^{(\mathrm{err})} = \min_{\{\boldsymbol{M}_0,\boldsymbol{M}_1\}} \left[q\ \mathrm{tr}(\boldsymbol{\Xi}_0\boldsymbol{M}_1) + (1-q)\ \mathrm{tr}(\boldsymbol{\Xi}_1\boldsymbol{M}_0)\right]. \tag{16}$$

Each POVM element can be parameterized as

$$\boldsymbol{M}_i(\boldsymbol{\lambda}_i) = \boldsymbol{U}_i(\boldsymbol{\lambda}_{\boldsymbol{U}_i})^\dagger \boldsymbol{D}_i(\boldsymbol{\lambda}_{\boldsymbol{D}_i})^2 \boldsymbol{U}_i(\boldsymbol{\lambda}_{\boldsymbol{U}_i}), \tag{17}$$

where

$$\boldsymbol{U} = e^{i\gamma/2}\begin{bmatrix} e^{i\mu}\cos\xi & e^{i\nu}\sin\xi \\ -e^{-i\nu}\sin\xi & e^{-i\mu}\cos\xi \end{bmatrix}. \tag{18}$$

Substituting $\boldsymbol{M}_1 = \boldsymbol{I} - \boldsymbol{M}_0$ yields a simplified optimization problem:

$$P_{\mathrm{err}} = q - \max_{\boldsymbol{\lambda}_0} \mathrm{tr}(\boldsymbol{\Delta}\boldsymbol{M}_0(\boldsymbol{\lambda}_0)), \tag{19}$$

where $\boldsymbol{\lambda}_0 = \{\boldsymbol{\lambda}_{\boldsymbol{U}_0}, \boldsymbol{\lambda}_{\boldsymbol{D}_0}\}$ are experimentally tunable parameters with $\boldsymbol{\lambda}_{\boldsymbol{U}_0} = \{\mu_0, \nu_0, \xi_0\}$ and $\boldsymbol{\lambda}_{\boldsymbol{D}_0} = \{\epsilon_0, \theta_0, \phi_0\}$.

For a given pair of qubit states $\boldsymbol{\Xi}_0, \boldsymbol{\Xi}_1$, the optimal linear-optical parameters realizing the Helstrom POVM are obtained by numerically minimizing Eq. (19). In practice, photon polarization states $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}_1$ are prepared, and the parameters $\boldsymbol{\lambda}_0$ are optimized to minimize the error probability. The resulting configuration of beam splitters, phase shifters, and rotators is then applied to the setup. When single photons are sequentially transmitted, detections at G_0 and G_1 correspond to $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}_1$, respectively, implementing a sequence of single-copy Helstrom measurements with linear optics.

C. Example

As an example, we simulate the discrimination of the qubit pair

$$\boldsymbol{\Xi}_\pm = \tfrac{1}{2}\left(\boldsymbol{I} \pm \kappa\boldsymbol{\sigma}_{\mathrm{x}}\right), \tag{20}$$

where $\boldsymbol{\sigma}_{\mathrm{x}}$ is the X-Pauli operator and κ quantifies the state mixedness. For $\kappa = 0.1$, Fig. 1 compares the simulated error exponents for the exact and tunable POVMs. Optimizing the tunable parameters yields POVMs that effectively realize

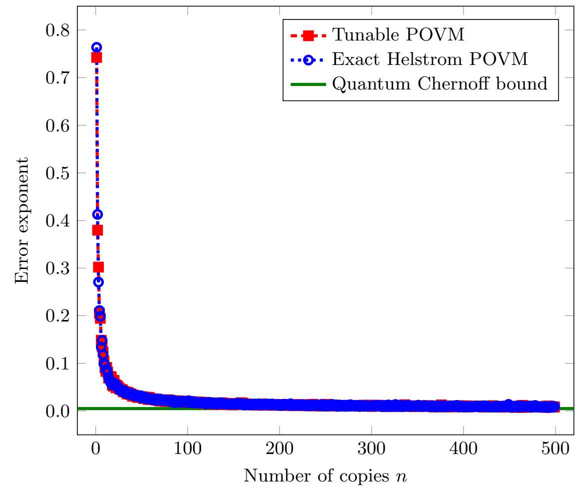

Fig. 1. Error exponents for repeated Helstrom measurements using the exact (3) and tunable POVMs (17).

the Helstrom measurement, and as the number of repetitions n increases, the empirical error exponent approaches the quantum Chernoff bound.

III. Conclusion

We proposed a theoretical scheme for implementing repeated Helstrom measurements on a tunable linear-optical platform. Numerical optimization of the optical parameters yields POVM operators realizing the single-copy Helstrom measurement. When repeated with majority voting, the method asymptotically approaches the quantum Chernoff bound, demonstrating that optimal discrimination can be achieved using only linear optics.

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