Entanglement Protection in Quantum Channels with Memory

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***Abstract*—Entanglement is an essential resource for several quantum communication and quantum information processing tasks. However, noise in the communication and information processing channels leads to degradation of entanglement and may cause entanglement sudden death. In particular, a maximally noisy channel can turn an entangled state into a separable one. Here we transmit a non-maximally entangled state through the maximally noisy amplitude damping (MNAD) channel with a tunable memory parameter. Our results show that the presence of channel memory facilitates the transmission of entanglement through MNAD. We follow the transmission by an entanglement purification stage to distill maximally entangled states, which can be utilized in various information processing protocols.**

I. Introduction

Many tasks in quantum communication and quantum information processing take advantage of entanglement to achieve their goal. However, entanglement of a state will degrade when sent through a noisy quantum channel. To address this problem, we can perform entanglement purification [1], [2] which obtains a small number of highly entangled states from a large number of copies of less entangled state. However, a maximally noisy quantum channel can render an entangled state to a separable one, making it impossible to perform any entanglement purification [3].

The majority of work on quantum channels has focused on memoryless quantum channels, which means that noise acts on qubits independently and identically [4]. Memoryless quantum channels provide a realistic description when the transmission rate is low. When the transmission rate is high, channel environment may retain memory of past events and thus memory effects need to be taken into account [4].

In this paper we transmit an entangled two-qubit state through an amplitude damping (AD) channel with memory or correlated amplitude damping (CAD) channel and show that even when the channel noise is maximal, the entanglement is not destroyed. Then we concentrate the entanglement of the channel output by implementing a purification protocol proposed in [5]. The memory in quantum channel improves the entanglement after purification process.

The remainder of this paper is organized as follows. In Section II, we review the purification protocol proposed in [5], which we will implement after transmitting a qubit through noisy CAD channel. Section III explains the correlated amplitude damping channel and implementation of purification protocol on channel output. We finally conclude in Section IV.

II. Entanglement Purification Protocol

Entanglement purification creates an ensemble of highly entangled qubit pairs out of larger ensemble with low fidelity, which is in range 0.5 and 1 [6]. The first QED protocol was proposed by Bennett *et al.* [1] and was further developed by Deutsch *et al.* [2]. Since these protocols work in a recursive way, they are called recurrence protocols. Entanglement swapping as entanglement purification was proposed by Bose *et al.* [7]. In this protocol, less entangled qubit state can be projected probabilistically to maximally entangled state or less entangled state. Purification protocol proposed in [5] uses entanglement swapping as a recurrence entanglement purification protocol. The protocol of the recurrence purification protocol with entanglement swapping is as follows:

1) Two pairs of qubits $|\phi\rangle_{AB_1} = \sqrt{a}\,|01\rangle + \sqrt{1-a}\,|10\rangle$ and $|\phi\rangle_{B_2C} = \sqrt{1-a}\,|01\rangle + \sqrt{a}\,|10\rangle$ are prepared.
2) $|\phi\rangle_{AB_1}$ is shared between Alice and Bob and $|\phi\rangle_{B_2C}$ is shared between Bob and Charlie through local individual AD channels.
3) *Entanglement Swapping*: Bob performs Bell state measurement in the basis

$$\left\{|\Phi\rangle^{\pm} = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right),\ |\Psi\rangle^{\pm} = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)\right\}.$$

If the measurement result is $|\Psi\rangle^{\pm}$, Alice and Charlie's state will collapse to mixed state

$$\rho_{AC} = \frac{1}{N}\left(2\gamma\bar{\gamma}^2 a\,|00\rangle\langle 00| + 2\bar{\gamma}^2 a\bar{a}\,|\Psi^{\pm}\rangle\langle\Psi^{\pm}|\right).$$

We call this a successful purification round for some a and noise severity γ parameter. If they got $|\Phi\rangle^{\pm}$ as measurement result, then purification process is aborted.
4) Two copies of ρ_{AC} are prepared through entanglement swapping. Alice and Charlie perform weak measurement $M_{\pm}$ on their qubits,

$$\begin{aligned} M_{+} &= \sqrt{b}\,|0\rangle\langle 0| + \sqrt{1-b}\,|1\rangle\langle 1| \\ M_{-} &= \sqrt{1-b}\,|0\rangle\langle 0| + \sqrt{b}\,|1\rangle\langle 1|, \end{aligned} \tag{1}$$

where b is the strength of the measurement. After performing this measurement, if measurement result of both parties on both pairs are M_{+}, Alice's state collapses into $\rho^{(a)}_{AC(M_w)} = \frac{1}{N}\left[2\gamma\bar{\gamma}ab\,|00\rangle\langle 00| + \bar{\gamma}^2\,|v\rangle\langle v|\right]$, where $|v\rangle = \sqrt{b}\,|01\rangle + \sqrt{1-b}\,|10\rangle$. Meanwhile Charlie's state

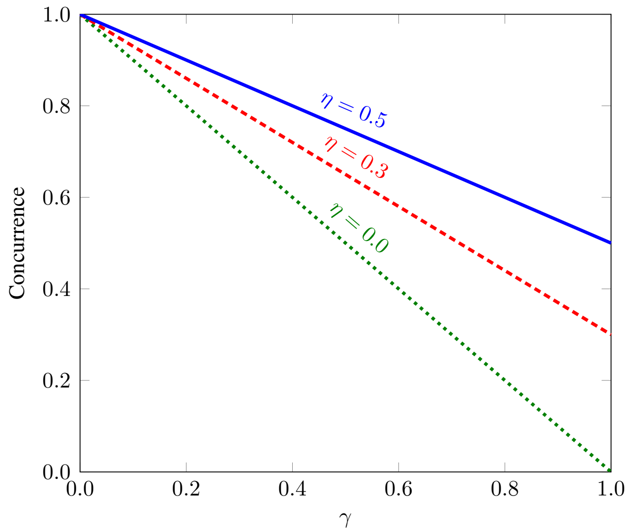

Fig. 1. Concurrence of $\rho_{AB_1(B_2C)}$ as a function of γ for $a = 0.5$ and $\eta = 0.0,\ 0.3,\ 0.5$.

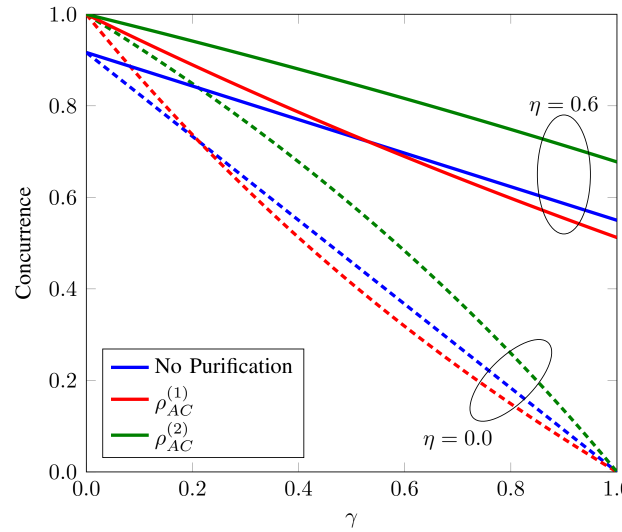

Fig. 2. Concurrence of ρ_{AB} (no purification), $\rho_{AC}^{(1)}$ (after the first round of purification) and $\rho_{AC}^{(2)}$ (after the second round of purification) for $a = 0.3$, $\eta = 0.0$ (dashed lines) and $\eta = 0.6$ (solid lines).

collapses into $\rho_{AC(M_w)}^{(c)}$ which has the same form as Alice but $|v\rangle = \sqrt{1-b}\,|01\rangle + \sqrt{b}\,|10\rangle$. If the measurement results are different, the purification process is aborted.

5) Use $\rho_{AC(M_w)}^{(a)}$ and $\rho_{AC(M_w)}^{(c)}$ as input state in step (3).

In this paper, qubit transmission through AD channel in step 2 will be modified with transmission through CAD channel. Here we assume that there is a correlation between Alice's channel and Bob's channel and Bob's channel and Charlie's channel.

III. Analysis on Correlated Amplitude Damping Channel

When a qubit is transmitted through CAD channel, it evolves as described by the following equation:

$$\mathcal{E}_{CAD}(\rho) = (1-\eta)\sum_{i,j=0}^{1} E_{ij}\rho E_{ij}^{\dagger} + \eta\sum_{k=0}^{1} A_k\rho A_k^{\dagger}, \tag{2}$$

where η is the memory paramter, E_{ij} $(i,j=0)$ are Kraus operators of AD channel and A_k $(k=0,1)$ are Kraus operators of fully correlated amplitude damping channel (FCAD). Memory paremeter $\eta \in [0,1]$ means that with probability η, the noise is correlated and with probability $(1-\eta)$, the noise is uncorrelated. Kraus operators of AD channel are given as

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix},\ E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}.$$

Kraus operators of FCAD channel are given as [8],

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma} \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 & \sqrt{\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

After going through CAD channel, state $|\phi\rangle_{AB_1}$ and $|\phi\rangle_{B_2C}$ transform into

$$\rho_{AB_1} = \begin{bmatrix} \gamma\bar{\eta} & 0 & 0 & 0 \\ 0 & a & \sqrt{a\bar{a}} & 0 \\ 0 & \sqrt{a\bar{a}} & \bar{a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\rho_{B_2C} = \begin{bmatrix} \gamma\bar{\eta} & 0 & 0 & 0 \\ 0 & \bar{a} & \sqrt{a\bar{a}} & 0 \\ 0 & \sqrt{a\bar{a}} & a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively, where $\bar{x}$ denotes $1-x$. After being sent to Alice, Bob and Charlie, we concentrate the entanglement by implementing the purification protocol stated above. To quantify the purification process, entanglement of quantum system before and after purification process is calculated using concurrence [9] : $C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where λ_i are eigenvalues of matrix $\rho\tilde{\rho}$, $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^{\dagger}(\sigma_y \otimes \sigma_y)$ in decreasing order. The simpler expression for this concurrence is [10]:

$$C(\rho) = 2\max\{0, \sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}}, \sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}}\}, \tag{3}$$

where ρ_{ij} is the ijth element of the density matrix ρ. After going through the CAD channel both initial state entanglement will decrease to,

$$C(\rho_{AB_1}) = C(\rho_{B_2C}) = 2(\eta\gamma + \bar{\gamma})\sqrt{a\bar{a}}. \tag{4}$$

If we set $\eta = 0$, (4) will turn into concurrence for AD channel $C(\rho_{AB_1}) = 2\gamma\sqrt{a\bar{a}}$ which is consistent with [5]. From (4), we can see that if $\gamma = 1$, the concurrence of state ρ_{AB_1} is $C(\rho_{AB_1}) = 2\eta\sqrt{a\bar{a}} \neq 0$, which means that the memory parameter η has effect on suppressing the decohorence of state $|\phi\rangle_{AB_1(B_2C)}$ when transmitted through CAD channel. We plotted concurrence of $|\phi\rangle_{AB_1(B_2C)}$ as a function of γ for $a = 0.5$ and $\eta = 0.0,\ 0.3$ and 0.5 in Fig. 1.

We can see that the stronger the memory effect, the smaller the channel noise effects the qubits.

Bob then performs Bell state measurement on his qubits. If the measurement result is $|\Psi^{\pm}\rangle$ then the state of Alice and Charlie will collapse to

$$\rho_{AC}^{(1)} = \frac{1}{N}\left(\alpha\,|00\rangle\,\langle 00| + \beta\,|\Psi^{\pm}\rangle\,\langle\Psi^{\pm}|\right), \tag{5}$$

where $\alpha = 2a\gamma\bar{\eta}\,(\eta\gamma+\bar{\gamma})$, $\beta = 2a\bar{a}\,(\eta\gamma+\bar{\gamma})^2$, and $N = 2a\,(\eta\gamma+\bar{\gamma})\,[\bar{a}\,(\eta\gamma+\bar{\gamma})+\gamma\bar{\eta}]$. The probability to obtain this state is $\frac{N}{2}$ and concurrence of ρ_{AC} is $C\left(\rho_{AC}^{(1)}\right) = \frac{B}{N}$, where superscript (n) denotes the nth round of purification protocol. The next step is Alice and Bob perform weak measurement on their qubits and use the qubits for second round of purification protocol.

In Fig. 2, we have plotted the concurrence after two round purification of two qubit state with $a = 0.3$ transmitted through CAD channel with $\eta = 0.0$ (dashed lines) and $\eta = 0.6$ (solid lines). When channel noise is 0.9, concurrence of qubit transmitted through channel with memory after second round purification is 5.3 times higher than qubit transmitted through memoryless channel.

IV. Conclusion

We studied about the effect of channel memory in CAD channel on entanglement purification protocol proposed in [5]. We found that the channel memory can suppress decoherence effects when transmitting $|\phi\rangle = \sqrt{a}\,|01\rangle + \sqrt{1-a}\,|10\rangle$ or $|\phi\rangle = \sqrt{1-a}\,|01\rangle + \sqrt{a}\,|10\rangle$ and initial entanglement can be concentrated by implementing protocol proposed in [5] even though the channel noise is maximal. The possible future work is implementing the protocol for other correlated channels and investigate the best channel to transmit our qubit.

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