On the Usefulness of Ancilla-Assisted Entanglement for Metrology

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***Abstract*—The presence of quantum noise during parameter estimation reduces precision in the estimate to sub-Heisenberg scaling. Ancilla-assisted bipartite entangled probes generally provide more precision than single qubit probes under such noisy dynamics. We consider the effect of phase-covariant noise models on the concurrence of an ancilla-assisted probe to identify the maximum noise strength for which the entanglement advantage is sustained over single qubit probe. Although the concurrence of the probe is nonzero for amplitude damping noise strength $\eta < 1$, but for depolarizing noise, the advantage of ancilla disappears even for nonmaximal noise strength p because the probe become separable.**

I. Introduction

The core purpose of quantum metrology is the precise and reliable estimation of unknown parameters. However, the presence of noisy dynamics reduces the reliability of the estimate. The efficiency with which we estimate the parameter then dictates the fidelity of the subsequent computations.

Entanglement has now and again proved itself to be a vital resource for gaining advantage over classical setups [1]–[4]. The primary measure of entanglement quantification for pure qubits is the concurrence proposed by [5] which was subsequently extended for the mixed state entanglement by [6]. The preservation of entanglement under noisy dynamics is significant for gaining the aforementioned advantage [7]

The recent advancements in metrology have investigated the idea of *entangled* ancilla-assisted probes for parameter estimation in noisy cases [8]. These probes utilize entanglement of the probe qubit with an ancilla qubit, the latter of which which does not undergo noisy parameter dynamics. It has been shown that the use of ancilla-assisted probe leads to an increase in the precision of the parameter estimate over uncorrelated probes under noisy dynamics for phase-covariant noise models [9].

In this paper, we show that the usefulness of ancilla-assisted entanglement is limited for the depolarizing channel for non-maximal noise strength. This is because the entanglement between the probe and ancilla vanishes for noise strength $p > 0.67$. This is in contrast to other phase-covariant noise models like amplitude damping and dephasing noise wherein some level of the entanglement is retained for nonmaximal noise strength.

II. Methods

For phase-covariant channels like amplitude damping and depolarizing channels commute with the phase encoding dynamics. Therefore, for phase-copvariant noise models, we can consider the scenario where noise acts before the parameter is encoded. The scenario discussed henceforth consists of one half of the Bell state undergoing noisy dynamics before being used as a probe in the parameter estimation. We see the effect of noise on the purity of the ancilla-assisted state and use this purity to observe the usefulness of ancilla-assisted parameter estimation under noisy dynamics. For this purpose, we first discuss the monotonicity of purity with noise strength and use it to identify the concurrence of the ancilla-assisted entangled state. For our analysis, we consider amplitude damping and depolarizing noise and show that although ancilla-assisted entangled state maybe useful for any level of noise strength, the same is not true for depolarizing noise. The noise type acting on the probe is identified *a* priori using quantum operation discrimination [10].

A. Purity measurement for noise strength estimate

We consider the Bell state of the form

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \tag{1}$$

in which one particle is the probe and the second is an ancilla. As the probe undergoes noisy dynamic map $\mathcal{N}$ the overall state evolves to the state

$$\rho_{\mathrm{o}} = \mathcal{N}\left(\left|\Phi^{+}\right\rangle\left\langle\Phi^{+}\right|\right). \tag{2}$$

To measure the purity of a quantum state, we perform the swap operator measurement defined as [11]

$$V = \sum_{j=1}^{4}\sum_{k=1}^{4} \left|\mu_j\right\rangle_{\rho_{\mathrm{o}}}\left\langle\mu_k\right| \otimes \left|\mu_k\right\rangle_{\rho_{\mathrm{o}}}\left\langle\mu_j\right|, \tag{3}$$

where the basis vector $\left|\mu_i\right\rangle,\ i \in \{k, j, l, m\}$, of the two-qubit Hilbert space can be chosen, e.g., the four Bell-state. Then, the purity of the state ρ_o, denoted by P_{o}, is obtained from the expectation of the swap operator V on two copies of the state ρ_{o} as

$$P_{\mathrm{o}} = \mathrm{Tr}\left(\left(\rho_{\mathrm{o}} \otimes \rho_{\mathrm{o}}\right) V\right) = \mathrm{Tr}\left(\rho_{\mathrm{o}}^2\right). \tag{4}$$

Note that the method for obtaining the purity has been provided by [12] using $\sqrt{V}$ gates on each of the two Bell state and an ancilla two qubit system. The ancilla-assisted state (1) with Alice's half under depolarizing noise has the purity [8]

$$P_{\mathrm{o}}\big|_{\mathrm{de}} = \mathrm{Tr}\left(\rho_{\mathrm{o}}^2\right) = \frac{3p^2}{4} - \frac{3p}{2} + 1. \tag{5}$$

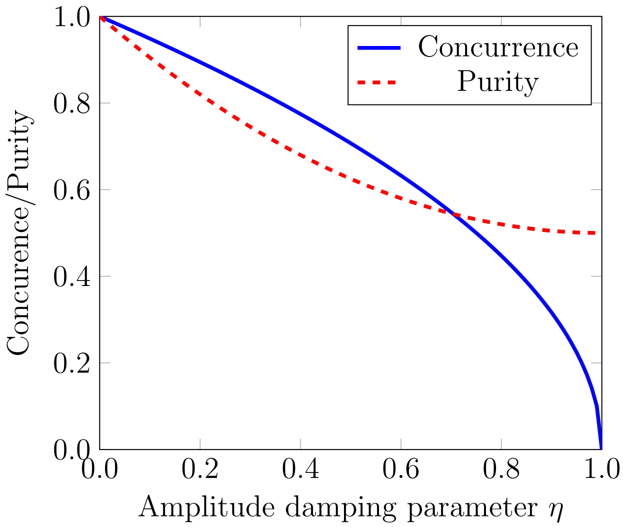

Fig. 1. Concurrence and purity of the ancilla-assisted state under amplitude damping dynamics.

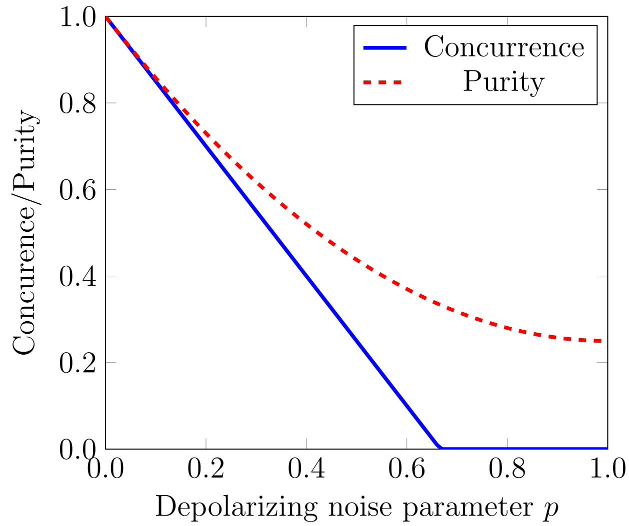

Fig. 2. Concurrence and purity of the ancilla-assisted state under depolarizing dynamics.

Whereas for the the amplitude damping noise, the purity of the bipartite state is

$$P_{\mathrm{o}}\big|_{\mathrm{ad}} = \mathrm{Tr}\left(\rho_{\mathrm{o}}^2\right) = 1 + \frac{\eta^2}{2} - \eta. \tag{6}$$

The two purity relations (5) and (6) are monotonic against the corresponding noise parameters [8]. This implies that by measuring P_{o}, we are able to estimate the noise strength. $\mathcal{O}$

B. Concurrence for Each DOF

For the input pure state of the form (1), the concurrence C is given by [5]

$$C(|\psi\rangle) = 2\left|\alpha(|\psi\rangle)\sqrt{1 - \{\alpha(|\psi\rangle)\}^2}\right|. \tag{7}$$

After the probe particle undergoes noisy dynamics, the evolved bipartite state ρ_{o}, has the concurrence C [6]

$$C = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{8}$$

where λ_i are the eigenvalues of $\rho_{\mathrm{o}}(\sigma_y \otimes \sigma_y)\rho_{\mathrm{o}}^*(\sigma_y \otimes \sigma_y)$ in decreasing order, where

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{9}$$

III. Results and Discussion

Fig. 1 and 2 provide the conservation of concurrence of the bipartite state as a function of noise strength. There is a stark difference between the concurrence in the two figures as the noise strength increases. Although the bipartite state retains some concurrence for the bipartite state for amplitude damping strength $\eta < 1$, the same is not true for depolarizing noise. For depolarizing noise, the concurrence is 0 for noise strength $p > 0.67$. Hence the advantage gained through the use of ancilla appears to be lost compared to a single qubit probe.

For future work, we will be looking to work with complex channel models with varying properties and see how the precision in the estimate and the entanglement of probe are correlated.

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