Hedge Maximum-Likelihood State Tomography

Ahmad Farooq and Hyundong Shin
Department of Electronic Engineering, Kyung Hee University, 17104, Korea
Email: hshin@khu.ac.kr

***Abstract*—Hedge maximum-likelihood estimation (HMLE) overcomes the problem of the rank deficient state which is present in maximum-likelihood estimation. HMLE is the quantum version of the "add β rule" for probability estimation. However, the parameter of β in HMLE is random. We provide a solution to approximate the parameter β. This enables the optimal estimation of rank deficient quantum states.**

I. Introduction

The fundamental building block of the quantum systems to represent the quantum information is a quantum state [1]–[3]. Quantum state tomography (QST) is a field that tells us how precisely we can tell the state by measuring it [4]–[6]. QST is the nondeterministic and classical estimation problem. Even if we have an ideal set of measurements, still we can not get the exact information about the quantum state. To gain knowledge about the true state, we require several numbers of identical copies of the same quantum state. Linear inversion is the first quantum state tomography algorithm which is used for the state estimation [7]. The problem associate with linear inversion is, it gives us sometimes a physically invalid state. To overcome this problem, the maximum-likelihood estimation (MLE) technique was introduced [8]. MLE always obtains the physical valid density matrix. It makes the negative eigenvalues to be zero which results in the rank deficient state.

The hedge maximum-likelihood estimation (HMLE) and Bayesian mean estimation (BME) [9] methods have been introduced as a substitute for MLE to resolve the rank deficient problem [10]. In this paper, We use the HMLE algorithm over the BME because of the low computational cost. HMLE is a simple algorithm to implement. The function which we maximize is the same as in MLE except for the extra hedging function. The hedging function involves the β parameter and the numerical value of β is completely random for the estimation of the quantum state. The performance of the HMLE depends on the parameter of β.

In this paper, we propose a method to approximate the parameter β for the estimation of the quantum state. We first use the MLE algorithm on measurement data and calculate the entropy of the estimated state from the MLE. We use the entropy value for the β parameter in HMLE. Our method reduces the overhead of adjusting the parameter of β in HMLE for any quantum state.

II. Hedge Maximum Likelihood

In this section, we mathematically formulate the HMLE. Any physical density matrix is defined as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{1}$$

where $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ and $\left|\psi_i\right\rangle$ are the eigenvectors.

The information we have about the physical density matrix is obtained through the measurement outcomes. The probability of getting these measurement outcomes are obtained by elementary projectors $\Pi_i = \left|\psi_i\right\rangle \left\langle\psi_i\right|$ through born's rule

$$p\left(i|\rho\right) = \operatorname{tr}\left(\Pi_i \rho\right). \tag{2}$$

The elementary projector forms the POVM. These POVM should sum to identity i.e.,

$$\sum_i \left|\psi_i\right\rangle \left\langle\psi_i\right| = \mathbb{I}. \tag{3}$$

We obtain the likelihood function as

$$\mathcal{L}\left(\rho\right) = \prod_{i=1}^{N} \operatorname{tr}\left(\Pi_i \rho\right)^{n_i}, \tag{4}$$

where n_i represents the number of times the measurement bases Π_i is observed.

The MLE maximize eq. (4) to estimate the desired density matrix. The estimated density matrix obtained from MLE can be rank deficient. It predicts the zero probability for some measurement vector $\left|\psi_i\right\rangle \left\langle\psi_i\right|$ such that

$$\left\langle\psi_i\right| \rho \left|\psi_i\right\rangle = 0. \tag{5}$$

Although it is physically possible but experimentally it can not be justified by the finite amount of the data. HMLE is an alternative of MLE to overcome this problem. The HMLE function is the product of Hedge function $\det\left(\rho\right)^{\beta}$ with the likelihood function

$$\mathcal{L}'\left(\rho\right) = \det\left(\rho\right)^{\beta} \mathcal{L}\left(\rho\right), \tag{6}$$

where β is a positive constant chosen randomly for the tomography of the quantum state. It should have the numerical value close to zero for the pure state while for the mixed state it should be $\frac{1}{2}$ or more. In tomography, we do not know the nature of the quantum state. We have to set the appropriate value of β parameter by using only the measurement results. We first employ the MLE technique on measurement outcomes

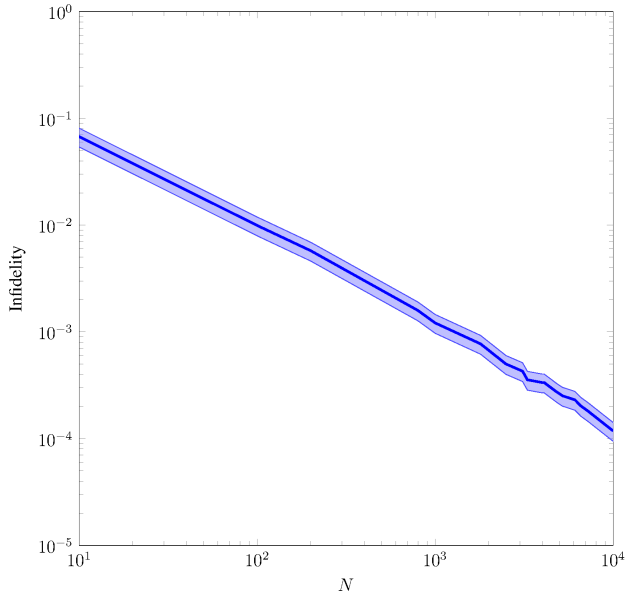

Fig. 1. Mean infidelity of the 100 randomly generated from haar measure qubit state of HMLE against the number of the copies is plotted. We take the 20 percent quantile of the mean. The decreasing curve shows that the β approximation is a good choice of the estimated density matrix.

and get the entropy of that state. The entropy of any quantum state is given by

$$S(\rho) = -\rho \log \rho = -\sum_i \lambda_i \log \lambda_i, \tag{7}$$

where λ_i are the eigenvalues of the density matrix.

For a maximally mixed state, the entropy of the density matrix is $\log_2 d$. It is zero for the pure state. We use the entropy value of the estimated density matrix from MLE for the β parameter. Then we apply the HMLE algorithm on the same measurement data to obtain the quantum state.

HMLE is inspired by Lidstone's Law. Suppose we observed N unknown sample from the identical and independent distribution of $\mathbf{p} = \{p_1, \cdots, p_k\}$ and have seen n_k. The natural MLE estimate gives us

$$p_k = \frac{n_k}{N}. \tag{8}$$

Some of the letters do not appear with this approach. We assign these letters with zero probability by this rule. This is perfectly fine when we have the probability truly zero. But usually, we have very small values of observing those letters. Add β rule overcomes this problem by

$$\hat{p_k} = \frac{n_k + \beta}{N + K\beta}. \tag{9}$$

This tells us that the lowest probability we have $\frac{\beta}{N}$.

III. Numerical Simulation

In this section, we will provide the numerical results of the qubit systems. We should select some common figures of merit to see the performance of the model. Infidelity is the most common and simple figure of merit. The infidelity of an estimated quantum state $\hat{\rho}$ from HMLE to true state ρ is given by

$$1 - F(\rho, \hat{\rho}) = 1 - \operatorname{tr}\sqrt{\sqrt{\rho}\hat{\rho}\sqrt{\rho}}. \tag{10}$$

We use the eigenvectors of the Pauli operators as basis measurements. These bases are called cubic measurements. The Pauli operators for the single-qubit system is defined as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{11}$$

We have randomly generated 100 states from the haar measure and plot the average infidelity of the quantum states obtained from the HMLE using the Pauli operator as a measurement basis against the number of iterations. The parameter β is approximated by entropy value first before using in HMLE. From Fig. 1, we can clearly see that the β is the good approximation as the infidelity of HMLE is decreasing when we increase the number of copies. The shaded region represents the 20% quantile of the mean line. HMLE achieves the same state as with the BME with less computational cost.

IV. Conclusion

In this paper, we have shown that the entropy of the estimated quantum state of MLE provides us a good approximate of β parameter used in the HMLE. In the future, we can improve the performance of our algorithm by using the diagonal basis to make the HMLE adaptive algorithm.

Acknowledgment

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2019R1A2C2007037).

References

[1] J. ur Rehman, A. Farooq, and H. Shin, "Discrete Weyl channels with Markovian memory," *IEEE J. Sel. Areas Commun.*, vol. 38, no. 3, pp. 413–426, Mar. 2020.

[2] U. Khalid, J. ur Rehman, and H. Shin, "Measurement-based quantum correlations for quantum information processing," *Sci. Rep.*, vol. 10, no. 1, p. 2443, Feb. 2020.

[3] J. ur Rehman, A. Farooq, Y. Jeong, and H. Shin, "Quantum channel discrimination without entanglement," *Quantum Inf. Process.*, vol. 17, no. 10, p. 271, Sep. 2018.

[4] M. A. Ullah, J. ur Rehman, and H. Shin, "Quantum frequency synchronization of distant clock oscillators," *Quantum Inf. Process.*, vol. 19, no. 5, p. 144, Mar. 2020.

[5] A. Farooq, J. ur Rehman, Y. Jeong, J. S. Kim, and H. Shin, "Tightening monogamy and polygamy inequalities of multiqubit entanglement," *Sci. Rep.*, vol. 9, p. 3314, Mar. 2019.

[6] A. Khan, J. ur Rehman, K. Wang, and H. Shin, "Unified monogamy relations of multipartite entanglement," *Sci. Rep.*, vol. 9, no. 1, p. 16419, Nov. 2019.

[7] T. Opatrný, D.-G. Welsch, and W. Vogel, "Least-squares inversion for density-matrix reconstruction," *Phys. Rev. A*, vol. 56, pp. 1788–1799, Sep. 1997.

[8] K. Banaszek, G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, "Maximum-likelihood estimation of the density matrix," *Phys. Rev. A*, vol. 61, p. 010304, Dec. 1999.

[9] R. Blume-Kohout, "Optimal, reliable estimation of quantum states," *New J. Phys.*, vol. 12, no. 4, p. 043034, Apr. 2010.

[10] ——, "Hedged maximum likelihood quantum state estimation," *Phys. Rev. Lett.*, vol. 105, no. 20, p. 200504, Nov. 2010.