Counterfactual Universal Logic Gates

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Abstract—**In this paper, we develop universal logic gates such as NAND and NOR gate using optical devices and chained quntuam Zeno (CQZ) effect which lead us to the counterfactual implementation of the universal logic gates without prior entanglement. We consider that the two binary inputs Alice (X) and Bob (Y); and the output Charlie (Z) of the countfactual logic gate are the distinct parties and demonstrate the counterfactual implementation of 2-inputs/1-output universal logic gates. Then, we generalize these results for n-inputs/1-output counterfactual universal logic gates.**

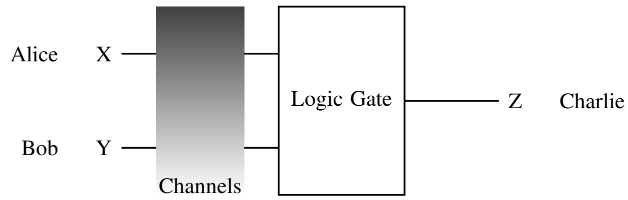

Fig. 1. **Basic model of logic gate.** We considered the three distinct parties Alice, Bob and Charlie where Alice and Bob equips the binary inputs X and Y of the logic gate and Charlie has the output Z.

I. Introduction

A logic gate is an electronic device to implement basic operation such as AND, OR, XOR, etc. and is the fundamental element of digital era of electronics, i.e., millions of logic gates are used in a microprocessor. Implementation of logic gates [1], [2] is an important concept in studying electronics since various laws and theorems of Boolean algebra can be implemented with a complete set of logic gates. Given two or more binary inputs, the logic gate carries out specified logical function and gives only one binary output.[1] Among the logic gates, the universal logic gates (NAND and NOR gates) [3], [4]–*the set of logic gates that can be used individually or together to realize all of the Boolean switching functions*– have their own importance in digital electronics. This property is called "Functional Completeness". Regarding this property, the entire microprocessor can be designed using only NAND or (and) NOR gates. In addition, the universal logic gates are also popular in designing Transistor-Transistor Logic (TTL) and Complementary Metal Oxide Semiconductor Transistor (CMOS) logics.

We consider the distributed computation model where the binary inputs of a 2-input logic gate are possessed by the remote parties say Alice (X) and Bob (Y) as shown in Fig. 1. In order to compute the output of a logic gate at third party Charlie (Z), Alice and Bob need to transmit their binary inputs to Charlie through a classical channel. In this article, we consider the same scenario and demonstrate the implementation of universal logic gates by using chained quantum Zeno (CQZ) gate but no physical particle is found in the transmission channel.

Counterfactual communication [5], [6] is a surprising phenomenon in quantum mechanics [7]–[10]. It takes account into the logic of CQZ gate and interaction free measurement [11], [12], thereby enabling the transfer of information between the sender and receiver but no physical particles traveling over the channel.[2] The CQZ gate transforms the vertically (V) polarized input photon as function of absorptive object as follows:

$$\begin{cases} |\mathrm{V}\rangle & \text{if AO} = 0, \\ |\mathrm{H}\rangle & \text{if AO} = 1, \end{cases} \tag{1}$$

where AO = 0 (1) denotes the absence (presence) of the absorptive object and $|\mathrm{V}(\mathrm{H})\rangle$ denotes the vertical (horizontal) polarized photon respectively. Recently, the counterfactual implementation of classical logic gates [15] has been demonstrated. Though the desired output of logic gate is achieved, but it increases the complexity of the circuit by using attenuators, phase shifters etc in addition to modified CQZ gate. In this paper, we demonstrate that CQZ gate itself works as logic gates. Our new scheme reduces the circuit complexity and enhances the probability of success of the protocol. The rest of the paper is structured as follows: first we demonstrate 2-input counterfactual universal logic (CUL) gates in Section II followed by n-input CUL gates in Section III.

II. Counterfactual Universal Logic Gates

Fig. 2 shows the schematics of counterfactual universal logic gates to implement a Boolean function where binary inputs X and Y are encoded as absence (0) or presence (1) of the absorptive objects. Here, the combination of CQZ gate and optical oscillators (OC) work as counterfactual logic gate unit. The Boolean function is implemented counterfactually on the two distinct inputs X and Y; and the output is generated at third party Charlie in terms of the polarization of the photon. Charlie starts the protocol by throwing his V polarized photon towards the OC_2 and inputs the photon in CQZ gate. In each inner cycle, the photon first interacts with Alice's absorptive object counterfactually followed by Bob's absorptive object. After

[1]Logic gates like NOT gate are 1-input/1-output gates.

[2]The CQZ gate is the nested version of the quantum Zeno gate [13] (see Fig. 2 in Reference [14]). The schematic of the CQZ is based on polarization of the input photon. Here we use only CQZ gate as the input photon is always V polarized. In the rest of the article, we refer CQZ gate as CQZ gate.

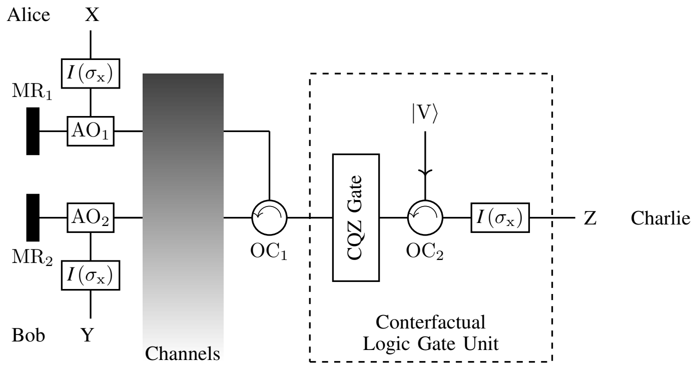

Fig. 2. **Schematics of counterfactual NOR (NAND) gate.** Charlie starts the protocol by throwing his V polarized photon towards the optical oscillator (OC) and inputs the photon in CQZ gate. Alice and Bob encode their binary inputs in the presence or absence of their respective absorptive object (AO). In each inner cycle, the photon first interacts with Alice's absorptive object followed by Bob's absorptive object. At the end of the protocol, Charlie applies $I\,(\sigma_x)$ operator on the output photon for NOR (NAND) gate where I is the single qubit identity operator and σ_x denotes the Pauli x operator, respectively.

M outer and N inner cycles, the polarization of the photon depends only on the value of X, Y and Boolean function.

A. Counterfactual NOR Gate

Table I shows the truth table of NOR gate corresponding to four possible binary inputs. The counterfactual implementation of all four cases is explained as follows:

- **Case 1: (0 0)**
  When both the binary inputs are low, the inner cycles of the CQZ gate act as blocking for the outer cycles. The state of the photon in mth $(\leq M)$ outer cycle is given as $|\psi_m\rangle_{00}$

$$|\psi_m\rangle_{00} = \cos^{m-1}\theta_M\left(\cos\theta_M|1\rangle + \sin\theta_M|0\rangle\right), \quad (2)$$

  where $\theta_M = \pi/(2M)$, $|\mathrm{H}\rangle = |0\rangle$ and $|\mathrm{V}\rangle = |1\rangle$. The state of the photon collapses back to the initial state after M outer cycles and the counterfactual NOR gate outputs $|1\rangle$, unless the photon is discarded in the counterfactual communication with probability

$$\lambda_{00} = \cos^{2M}\theta_M, \quad (3)$$

  tending to 1 as $M \to \infty$.
- **Case 2: (0 1)**
  For X=0 and Y=1, Alice allows the photon to pass but Bob blocks the transmission channel by introducing an absorptive object. In this case, the inner cycles act as non-blocking for outer cycles and each outer cycle rotates the state of the photon by an angle of θ_M. After m outer cycles, the CQZ gate transforms the state of the photon to

$$|\psi_m\rangle_{01} = \cos(m\theta_M)|1\rangle + \sin(m\theta_M)|0\rangle. \quad (4)$$

TABLE I
TRUTH TABLE OF UNIVERSAL LOGIC GATES

Cases	Alice X	Bob Y	Charlie NOR $Z=\overline{X+Y}$	Charlie NAND $Z=\overline{X\cdot Y}$
Case 1	0	0	1	1
Case 2	0	1	0	1
Case 2	1	0	0	1
Case 2	1	1	0	0

  Unless the photon is absorbed by the Charlie's absorptive object with probability

$$\lambda_{01} = \prod_{i=1}^{M}\left[1 - \sin^2(i\theta_M)\sin^2\theta_N\right]^N. \quad (5)$$

  where $\theta_N = \pi/(2N)$, the counterfactual NOR gate outputs $|0\rangle$ after M outer cycles.
- **Case 3: (1 0) & Case 4: (1 1)**
  As the photon first interacts with Alice's absorptive object (counterfactually), if X=1, the inner cycles are non-blocking for outer cycles regardless the state of Y. The state evolution of the photon is similar to case 2 where $|\psi_m\rangle_{1\mathrm{Y}} = |\psi_m\rangle_{01}$. The output of the counterfactual NOR gate is $|0\rangle$ for x=1 (independent of y) unless the photon is absorbed by the Alice's absorptive object with probability $\lambda_{1\mathrm{Y}} = \lambda_{01}$.

From the above discussion, we conclude that the counterfactual NOR gate outputs $|1\rangle$ if both the binary inputs are Low and $|0\rangle$ vice versa with certainty under the asymptotic limits of M and N. For the finite values of M and N, the average probability that the photon is not discarded is

$$P = \sum_{\mathrm{X}=0}^{1}\sum_{\mathrm{Y}=0}^{1} p_{\mathrm{XY}}\lambda_{\mathrm{XY}}, \quad (6)$$

where p_{XY} is the probability mass function of $\{00, 01, 10, 11\}$.

B. Counterfactual NAND Gate

In the previous subsection, we briefly explained the working of the counterfactual NOR gate with two binary inputs. Two-inputs counterfactual NAND gate works similar as counterfactual NOR gate. The only difference is that Alice and Bob need to apply σ_x operator before and after the counterfactual logic gate unit on their respective binary inputs. At the end of the protocol, Charlie simply applies σ_x operation on his output photon to complete the counterfactual NAND gate operation. Due to σ_x operation before the counterfactual logic gate unit, the state evolution of the photon corresponding to the inputs X=Y=i differ from counterfactual NOR gate where $i \in \{0, 1\}$.

$$|\phi_m\rangle_{00} = |\psi_m\rangle_{11}, \quad (7)$$

$$|\phi_m\rangle_{11} = |\psi_m\rangle_{00}, \quad (8)$$

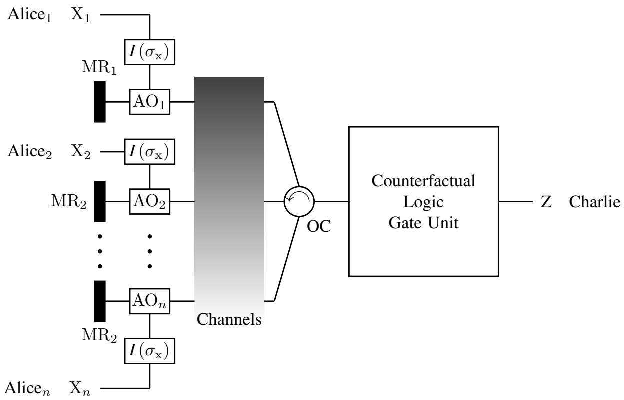

Fig. 3. **n-inputs/1-output counterfactual NOR (NAND) gate.**

unless the photon is discarded in the counterfactual logic gate unit with probability

$$\zeta_{00} = \lambda_{11}, \tag{9}$$
$$\zeta_{11} = \lambda_{00}. \tag{10}$$

III. n-INPUTS/1-OUTPUT COUNTERFACTUAL UNIVERSAL LOGIC GATES

Without a loss of generality, we consider the n-inputs logic gate as shown in Fig. 3. Similar to 2-inputs counterfactual logic gates, the photon interacts with each absorptive object in every inner cycle. For n-inputs counterfactual NOR gate, the state evolution of the photon after $m\,(\leq M)$ is given as

$$|\chi_m\rangle_{00\ldots0} = |\psi_m\rangle_{00}\,, \tag{11}$$
$$|\chi_m\rangle_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n\neq 00\ldots0} = |\psi_m\rangle_{11}\,. \tag{12}$$

unless the photon is discarded in the counterfactual logic gate unit with probability

$$\gamma_{00\ldots0} = \lambda_{00}, \tag{13}$$
$$\gamma_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n\neq 00..0} = \lambda_{11}. \tag{14}$$

In case the NAND gate is applied, the state transformation of the input photon and the probabilities that the photon is not discarded are given as

$$|\chi_m\rangle_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n\neq 11\ldots1} = |\phi_m\rangle_{00}\,, \tag{15}$$
$$|\chi_m\rangle_{11..1} = |\phi_m\rangle_{11}\,, \tag{16}$$
$$\gamma_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n\neq 11..1} = \zeta_{00}, \tag{17}$$
$$\gamma_{11\ldots1} = \zeta_{11}. \tag{18}$$

Here it is important to note that the success probability of the coutnerfactual implementation of the NAND and NOR gates for the given input is independent of n. In Fig. 4, we plot the average probability of success for $n = 2$ under the uniform probability distribution of possible inputs. Surprisingly, the average success probability for $n = 2$ under the uniform distribution of $\{00, 01, 10, 11\}$ is same for NAND and NOR gate.

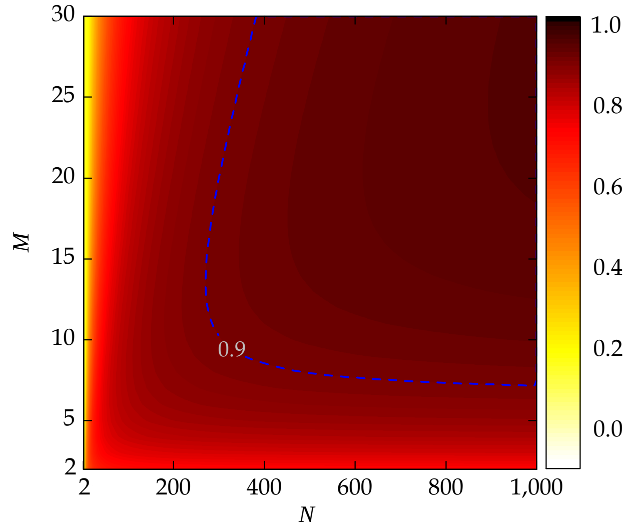

Fig. 4. Average success probability for $n = 2$ under the uniform distribution of possible inputs as function of M and N.

IV. CONCLUSIONS

In this paper, we presented a new scheme of n-inputs/1-output counterfactual universal logic gates by using CQZ gate. For $n = 2$, Alice and Bob can independently control the absorptive object and hence determining Charlie's output without any physical particles traveling over the channel. Though the process involves three parties, there is no need for preshared entanglement. Based on this gate design, it can be extended to build other counterfactual logic gates and thereby using the combination of these gates will help to prepare the implementation of other complex circuits.

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