Optimal Parameter Estimation
of Noisy Pauli Channels

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Abstract—We analyze the entanglement-based optimal parameter estimation protocol of Pauli channel under noise. Specifically, we consider that the prepared probes undergo depolarizing noise before going through the parameter estimation protocol. This scenario introduces a bias in the otherwise unbiased estimator. Then, we propose a bias mitigation method which provides same scaling as that of optimal method, except for a constant multiplicative factor, which is a function of initial noise strength. However, bias mitigation does not always provide an advantage over the biased technique in terms of MSE. We identify and discuss the regions where bias mitigation worsens and improves the MSE performance.

I. INTRODUCTION

Efficient practical implementations of quantum information processing (QIP) protocols generally require a complete knowledge of all system components [1]–[5]. This includes the knowledge about the quality of preparation of the states input to the system and the closeness of actual system operation to the intended operations. There exist different approaches to obtain such knowledge about the system components. The most common and the oldest of these approaches is the tomography. Other methods include randomized benchmarking and quantum gate set tomography.

Tomography of quantum systems is the method of constructing the numerical description of an unknown quantum system by performing several rounds of preparation and measurement of the system-of-interest. For example, the purpose of state tomography of a quantum state is to obtain the density matrix of an unknown quantum state. Likewise, quantum process tomography aims to obtain the numerical description of an unknown quantum process. The numerical description of a quantum channel can be either in the form of Choi state of the channel or finding the unknown parameters of a parametric family of quantum channels.

Pauli channels constitute an important class of parametric qubit channels relevant to practical implementations of several QIP tasks. Special classes of these channels include identity, bit-flip, phase-flip, two-Pauli, and qubit depolarizing channels. Pauli qubit channels have also been generalized to d-dimensional Hilbert spaces [6]. Due to their importance, Pauli channels and their generalizations have been extensively discussed in the literature [7]–[13]. Several researchers have proposed different methods for finding the parameters of a physically available but unknown Pauli channel, i.e., process tomography of Pauli qubit channels and their generalizations.

In this paper we analyze the method of quantum parameter estimation of generalized Pauli channels (QPEC) proposed in [14] under noisy probes. That is, we attempt to execute QPEC when the probe states undergo another noisy evolution before going through the actual channel of interest. We consider the initial noise to be the depolarizing noise of specific strength. We analyze the performance of QPEC in this scenario in terms of mean square error (MSE) and its dependence on the number of tomography iterations (N). We find that MSE vs N curve exactly follows the noiseless case for some $N \leq N_{\mathrm{id}}$. Afterwards, there is an intermediate region for $N_{\mathrm{id}} \leq N \leq N_{\mathrm{in}}$ where improvement in MSE as a function of increasing N is smaller than the previous region. Finally, $N \geq N_{\mathrm{in}}$ region provides no improvement in MSE by increasing N. Therefore, the MSE of the QPEC is strictly limited by the preparation quality of the probe states imposing a hard limit on N beyond which increasing N simply wastes the channel iterations without providing any tomographical advantage.

The remainder of this paper is organized as follows. In Section II we outline the method of parameter estimation of Pauli channels. Then, in Section III we analyze the performance of this method when the probe states undergo a depolarizing channel before being input to the QPEC. We finally conclude in Section IV.

II. PARAMETER ESTIMATION OF PAULI CHANNELS

A Pauli qubit channel is defined by the mapping on two-dimensional Hilbert space $\mathcal{H}_2$ as

$$\mathcal{N}_{\mathrm{P}}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^{\dagger}, \qquad (1)$$

where σ_0 is the identity on $\mathcal{H}_2$, σ_1, σ_2, and σ_3 are Pauli X, Y, and Z matrices, respectively, and $p = [p_0, p_1, p_2, p_3]$ is a probability vector. A Pauli qubit channel can be generalized to a d-dimensional Hilbert space $\mathcal{H}_d$ as

$$\mathcal{N}_{\mathrm{GP}}(\rho) = \sum_{i,j=0}^{d-1} p_{i,j} \sigma_{i,j} \rho \sigma_{i,j}^{\dagger}, \qquad (2)$$

where $p = [p_{i,j}]_{0 \leq i,j \leq d-1}$ is a d^2-dimensional probability vector, and $\sigma_{i,j}$ form an orthonormal basis of d-dimensional density matrices, i.e., any density matrix on $\mathcal{H}_d$ can be written in the form

$$\rho = \frac{1}{d} \sum_{i,j=0}^{d-1} r_{i,j} \sigma_{i,j},$$

where $r = [r_{i,j}]_{0 \le i,j \le d-1}$ is the Bloch vector for ρ with components $r_{i,j} = \text{tr}(\rho \sigma_{i,j}) \in \mathbb{C}$. The orthonormality condition is

$$\text{tr}\left(\sigma_{i,j}^{\dagger} \sigma_{k,\ell}\right) = d\delta_{i,k}\delta_{j,\ell},$$

with $\delta_{i,j}$ being the Kronecker's delta function which is equal to 1 if $i = j$, and is zero otherwise.

QPEC of [14] can be simply described as follows. Prepare N copies of the maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i\rangle \otimes |f_i\rangle,$$

where $|e_i\rangle$ is the standard orthonormal (computational) basis and $|f_i\rangle$ is the eigenbasis of one of $\sigma_{i,j}$. Then, let the channel act on the second part of this state, i.e.,

$$\rho_{\text{GP}} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i\rangle \langle e_j| \otimes \mathcal{N}_{\text{GP}}\left(|f_i\rangle \langle f_j|\right)$$

$$= \sum_{i,j=0}^{d-1} p_{i,j} |\psi_{i,j}\rangle \langle \psi_{i,j}|,$$

where

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |e_k\rangle \otimes \sigma_{i,j} |f_k\rangle.$$

It is simple to show that $\{|\psi_{i,j}\rangle\}_{i,j}$ form an orthonormal basis of $\mathcal{H}_d \otimes \mathcal{H}_d$, thus decomposing ρ_{GP} into its eigenbasis. Then, N copies of ρ_{GP} are measured in the basis $\{|\psi_{i,j}\rangle\}_{i,j}$ and the estimates $\{\hat{p}_{i,j}\}_{i,j}$ on the channel parameters $\{p_{i,j}\}_{i,j}$ are obtained as

$$\hat{p}_{i,j} = \frac{n_{i,j}}{N},$$

where $n_{i,j}$ is the number of times measurement result corresponding to $|\psi_{i,j}\rangle$ was obtained.

From the Born's rule, it is simple to verify that the estimator as defined above is unbiased, i.e.,

$$\hat{p}_{i,j} = \text{tr}\left(\rho_{\text{GP}} |\psi_{i,j}\rangle \langle \psi_{i,j}|\right) = p_{i,j}.$$

It was proven in [14] that this is the optimal strategy and no additional entanglement can improve the estimation precision.

III. PARAMETER ESTIMATION WITH NOISE

In this Section we assume that the probe states undergo a *global* depolarizing noise of strength λ before going into the QPEC protocol. That is,

$$\rho_{\text{dep}}^{\lambda} = (1-\lambda)|\psi\rangle\langle\psi| + \lambda\pi,$$

are input to the QPEC protocol, where $\pi = I/d^2$ is the maximally mixed state on $\mathcal{H}_d \otimes \mathcal{H}_d$. Then, from the linearity of quantum channels

$$\rho_{\text{GP}}^{\text{dep}} = (1-\lambda) \sum_{i,j=0}^{d-1} p_{i,j} |\psi_{i,j}\rangle \langle \psi_{i,j}| + \lambda\pi.$$

Then, the same estimation strategy yields

$$\hat{q}_{i,j} = \text{tr}\left(\rho_{\text{GP}} |\psi_{i,j}\rangle \langle \psi_{i,j}|\right) = (1-\lambda)p_{i,j} + \frac{\lambda}{d^2} \ne p_{i,j}.$$
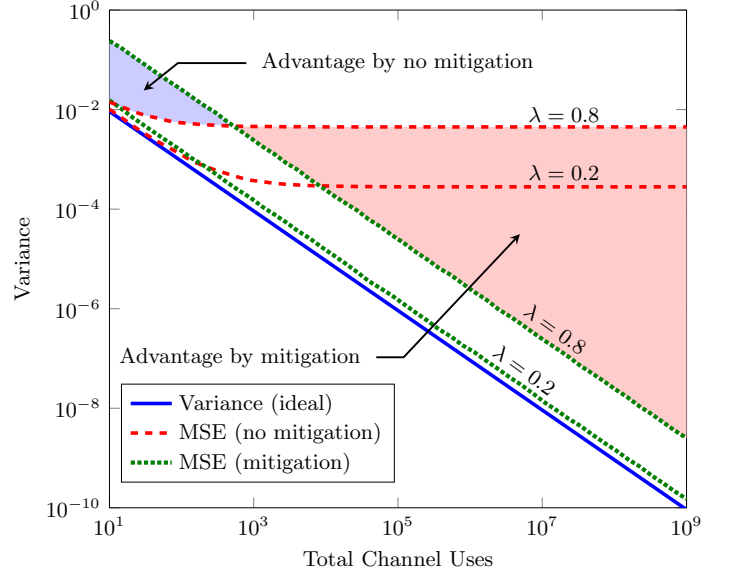

Fig. 1. Performance of QPEC under noise with and without bias mitigation for $d = 3$ and a moderately noisy channel.

Therefore, the estimation strategy of the previous section yields biased estimates. However, if the noise strength λ is known, we can recover the unbiased estimates on $p_{i,j}$ as

$$\hat{p}_{i,j} = \frac{\hat{q}_{i,j}}{1-\lambda} - \frac{\lambda}{(1-\lambda)d^2}. \tag{3}$$

In the following, we numerically investigate the performance of this modified estimator for the two cases of known and unknown strength of depolarizing noise.

Fig. 1 shows the performance of QPEC under noise with and without bias mitigation. The generalized Pauli channel that we use for this numerical example is a discrete Weyl channel and obtained by setting $\gamma = 0.5, d = 3$ in channel obtaining method of [15, Section IV] and is a moderately noisy channel. It can be seen that the MSE of QPEC under noise becomes independent of increasing channel iterations N when N is large. On the other hand, bias mitigation allows to obtain MSE with same scaling of noiseless case, except for a constant multiplicative factor, which is a function of noise strength λ.

Furthermore, we can identify two distinct regions based on the performance comparison of the two techniques, QPEC with and without bias mitigation. In Fig. 1, region shaded with blue color shows the advantage of no mitigation. As we increase N, the performance gap between the QPEC and the bias mitigated QPEC continues to shrink, before reaching the point where both schemes provide same MSE performance. Then, as we further increase N, the bias mitigation strategy begins to provides the advantage and continues to do so for increasing N. For the sake of clarity, we have highlighted these two regions only for $\lambda = 0.8$ case. However, these two distinct regions can be easily recognized in the curves for $\lambda = 0.2$ case as well. In light of these distinct regions, optimal method for the QPEC can be a hybrid approach where bias mitigation is not utilized for small N and utilized when N is large.

IV. Conclusions

We have showed the presence of a bias in the estimates of QPEC under depolarizing noise. This noise and the introduced bias highly impacts the MSE performance of QPEC estimators in the high channel uses (large N) regime. In the case when the noise strength is known, we introduced the bias mitigation method which not only removes the bias but also recovers the ideal scaling of estimates as a function of N, except with a constant multiplicative factor. In the future it is worth investigating if similar noise/bias mitigation schemes can be devised for arbitrary type of noise, and require no/partial knowledge of noise strength.

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