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The Three-Dimensional Partial Differential Equation with Constant Coefficients of Time-Delay of Alternating Direction Implicit Format

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Abstract

In this paper, we consider the delay partial differential equation of three dimensions with constant coefficients. We established the alternating direction difference scheme by the standard finite difference method, gave the order of convergence of the format and the expression of the difference scheme truncation errors.

Keywords

Alternating Direction Implicit Format, Stability, Time-Delay, Partial Differential Equation, Three-Dimensional

1. Introduction

The delay partial differential equations are often encountered in the study of heat conduction, gas diffusion and power engineering. The existence of time delay brings great difficulties to the research and numerical solution of delay equations. If the delay term has a little effect on the system, we often ignore delay term, and replace ordinary (partial) differential equation with the time-delay ordinary (partial) differential [1]. And then, we can solve the problem by establishing Euler scheme, Richardson scheme, and Crank-Nicolson scheme, etc. [2]. Sometimes, a small time-delay, like launch satellite, spacecraft control, missile guidance, can have a great impact on the system, and even 1%-second time delay [3]. From this, the delay must not be neglected.

In general, the exact solution of time-delay differential equation is difficult to project. In practical applications, the numerical methods are often used to obtain approximate solutions. Therefore, the numerical analysis of the time-delay differential equation is especially important. It has become an important part of the field of computational mathematics. Among the many numerical methods, finite difference method has been widely used for simple structure and easy processing. At present, the traditional finite difference method for the partial differential equations of time delay is primarily used [4-6]. In 1955, Peaceman and Rachford [7] first introduced the alternating direction method which is called $P-R$ format. The alternating direction method can transform the high-dimensional problem into a series of one-dimensional problems to solve, and reduce the computational workload. This format is

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unconditional stability, and has second order accuracy. And then the Douglas' format [8] came into being, this scheme has a second order truncation error and is suitable for three dimensional cases. A compact alternating direction format is established for the initial boundary value problem of two-dimensional constant coefficient parabolic equations by Sun and Li [9], the solvability, stability and convergence are studied. This format can also be applied to time-delay equations.

In 1998, Lu [10] proposed the monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays. They derived two modified iterative schemes by combing the method of upper-lower solutions and the Jacobi method or the Gauss-Seidel method, and discussed the convergence and stability of the monotone iterative schemes.

$$
\begin{aligned}
&u_t - D\Delta u = f(x,t,u,u(x,t-\tau)), \qquad (x,t)\in\Omega\times(0,T], \\
&\alpha(x,t)u_v + \beta(x,t)u = h(x,t), \qquad (x,t)\in\partial\Omega\times(0,T], \\
&u(x,t) = u_0(x,t), \qquad (x,t)\in\Omega\times[-\tau,0], \qquad \Omega\subset R^p .
\end{aligned} \tag{1}
$$

where T, τ are positive constants with $T > \tau > 0$. Ω is a bounded domain in R^p. We assume that $D = D(x,t) > 0$ and $h(x,t), u_0(x,t)$ are both Hölder continuous. The function $f(x,t,u,\eta)$ is a C^1-function of u, η and is monotone in η.

In 2002, Jiang et al. [11] gave the classical format and obtained the convergence for the initial boundary value problem of neutral parabolic equation with time delay, where τ is a positive constant.

$$
\begin{aligned}
&u_t(x,t) - bu_t(x,t-\tau) = u_{xx}(x,t) + f(x,t), \ (x,t)\in(0,L)\times(0,T], \\
&u(x,t) = \phi(x,t), \qquad (x,t)\in(0,L)\times[-\tau,0], \\
&u(0,t) = u(L,t) = 0, \qquad t\in[-\tau,T].
\end{aligned} \tag{2}
$$

In 2011, Liu et al. [12] used the Wavelet collocation method to obtain higher numerical accuracy for initial value problems of the following equation with time delay, where $\tau_i > 0, \tau_{\max} = \max\{\tau_i\}$.

$$
\begin{aligned}
&y'(t) = ay(t) + \sum_{i=0}^{m} b_i y(t-\tau_i) + f(t), \ t\in[0,1], \\
&y(t) = \psi(t), \qquad t\in[-\tau_{\max}, 0],
\end{aligned} \tag{3}
$$

Until now, there is no research about the three-dimensional partial differential equation of time-delay with constant coefficients; nevertheless, this kind of equation is widely used in mathematical model. Accordingly, it is obvious to solve the numerical scheme of the equation. In this paper, we consider the following problems:

$$u_t - au_{xx} - bu_{yy} - cu_{zz} = uf(x,y,t,u_\tau), \ (x,y,z,t)\in\Omega\times(0,T], \tag{4}$$

$$u(x,y,z,0) = \varphi(x,y,z), \ (x,y,z)\in\overline{\Omega}, \tag{5}$$

$$u(x,y,z) = \phi(x,y,z), \ (x,y,z)\in\partial\Omega, \ 0 < t \le T, \tag{6}$$

where $\Omega = (0,1)\times(0,1)$, $\partial\Omega$ is Ω border, $\overline{\Omega} = [0,1]\times[0,1]$, a, b, c, T, τ are the set of real numbers, $f \geq 0$ and φ, ϕ are the given smooth function, $u_\tau = u(x, y, z, t-\tau)$ is delay term.

2. Notations and Lemmas

Let m, n, p be the positive integers, $\Delta t = T/n$, $\tau = p\Delta t$, the time domain $[0,T]$ is covered by $\Omega_{\Delta t} = \{t_k = k\Delta t \mid 0 \leq k \leq n\}$. For a positive integer m, let $h = 1/m$ be the step length of spatial approximation, and

$$x_i = ih,\ 0 \leq i \leq m,\ y_j = jh,\ 0 \leq j \leq m,\ z_l = lh,\ 0 \leq l \leq m,\ \Omega_h = \{(x_i, y_j, z_l) \mid 0 \leq i, j, l \leq m\}.$$

$$\gamma = \{(0,l),(m,l) \mid 0 \leq l \leq m\} \cup \{(0,j),(m,j) \mid 0 \leq j \leq m\} \cup \{(i,0),(i,m) \mid 1 \leq i \leq m-1\},$$

$$t_{k+1/2} = \frac{1}{2}(t_k + t_{k+1}),\ f_{ij}^{k+1/2} = f(x_i, y_j, z_l, t_{k+1/2}, u_\tau).$$

For any grid function $v = \{v_{ijl}^k \mid 0 \leq i, j, l \leq m, 0 \leq k \leq n\}$ is defined on $\Omega_h \times \Omega_{\Delta t}$, considering the following notations:

$$v_{ijl}^{k+1/2} = \frac{1}{2}(v_{ijl}^k + v_{ijl}^{k+1}),\ \delta_t v_{ijl}^{k+1/2} = \frac{1}{\tau}(v_{ijl}^{k+1} - v_{ijl}^k),$$

$$\delta_x v_{i+1/2,j,l}^k = \frac{1}{h}(v_{i+1,j,l}^k - v_{ijl}^k),\ \delta_y v_{i,j+1/2,l}^k = \frac{1}{h}(v_{i,j+1,l}^k - v_{ijl}^k),$$

$$\delta_z v_{i,j,l+1/2}^k = \frac{1}{h}(v_{i,j,l+1}^k - v_{ijl}^k),\ \delta_x^2 v_{ijl}^k = \frac{1}{h}(\delta_x v_{i+1/2,j,l}^k - \delta_x v_{i-1/2,j,l}^k),$$

$$\delta_y^2 v_{ijl}^k = \frac{1}{h}(\delta_y v_{i,j+1/2,l}^k - \delta_y v_{i,j-1/2,l}^k),\ \delta_z^2 v_{ijl}^k = \frac{1}{h}(\delta_z v_{i,j,l+1/2}^k - \delta_z v_{i,j,l-1/2}^k).$$

3. Construction and Error of the Difference Scheme

Suppose $U = \{U_{ijl}^k \mid 0 \leq i, j, l \leq m, 0 \leq k \leq n\}$, where $U_{ijl}^k = u(x_i, y_j, z_l, t_k),\ 0 \leq i, j, l \leq m,\ 0 \leq k \leq n$.

Consider equation (4) at the point $(x_i, y_j, z_l, t_{k+1/2})$, we obtain

$$\begin{aligned} &u_t(x_i, y_j, z_l, t_{k+1/2}) - au_{xx}(x_i, y_j, z_l, t_{k+1/2}) - bu_{yy}(x_i, y_j, z_l, t_{k+1/2}) - cu_{zz}(x_i, y_j, z_l, t_{k+1/2}) \\ &= u(x_i, y_j, z_l, t_{k+1/2}) f_{ijl}^{k+1/2}, \qquad 1 \leq i, j, z \leq m-1,\ 0 \leq k \leq n-1. \end{aligned} \tag{7}$$

Using the Taylor expansion, we can get the truncation errors,

$$u_t(x_i, y_j, z_l, t_{k+1/2}) = \delta_t U_{ijl}^{k+1/2},\ u_{xx}(x_i, y_j, z_l, t_{k+1/2}) = \frac{1}{2}(\delta_x^2 U_{ijl}^k + \delta_x^2 U_{ijl}^{k+1}),$$

$$u_{yy}(x_i, y_j, z_l, t_{k+1/2}) = \frac{1}{2}(\delta_y^2 U_{ijl}^k + \delta_y^2 U_{ijl}^{k+1}),\ u_{zz}(x_i, y_j, z_l, t_{k+1/2}) = \frac{1}{2}(\delta_z^2 U_{ijl}^k + \delta_z^2 U_{ijl}^{k+1}). \tag{8}$$

Substituting them into equation (7), we obtain

$$\delta_t U_{ijl}^{k+1/2} - a\delta_x^2 U_{ijl}^{k+1/2} - b\delta_y^2 U_{ijl}^{k+1/2} - c\delta_z^2 U_{ijl}^{k+1/2} = U_{ijl}^{k+1/2} f_{ijl}^{k+1/2},\ 1 \le i, j, l \le m-1,\ 0 \le k \le n-1 .$$

Then

$$\delta_t U_{ijl}^{k+1/2} - a\delta_x^2 U_{ijl}^{k+1/2} - b\delta_y^2 U_{ijl}^{k+1/2} - c\delta_z^2 U_{ijl}^{k+1/2} + \frac{ab}{4}\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{k+1/2} + \frac{bc}{4}\tau^2\delta_y^2\delta_z^2\delta t U_{ijl}^{k+1/2} + R_{ijl}^{k+1/2}$$

$$+\frac{ac}{4}\tau^2\delta_x^2\delta_z^2\delta_t U_{ijl}^{k+1/2} - \frac{abc}{4}\tau^3\delta_x^2\delta_y^2\delta_z^2 U_{ijl}^{k+1/2} = U_{ijl}^{k+1/2} f_{ijl}^{k+1/2},\ 1 \le i, j, l \le m-1,\ 0 \le k \le n-1 \tag{9}$$

$$R_{ijl}^{k+1/2} = [\frac{1}{24}u_{ttt}(x_i, y_j, z_l, \eta_{ijl}^k) - \frac{1}{8}u_{xxtt}(x_i, y_j, z_l, \overline{\eta}_{ijl}^k) - \frac{1}{8}u_{yytt}(x_i, y_j, z_l, \overline{\overline{\eta}}_{ijl}^k) - \frac{1}{8}u_{zztt}(x_i, y_j, z_l, \overline{\overline{\overline{\eta}}}_{ijl}^k)]\Delta t^2$$

$$-\frac{1}{24}[u_{xxxx}(\overline{\xi}_{ijl}^k, y_j, z_l, t_k) + u_{xxxx}(\overline{\xi}_{ijl}^{k+1}, y_j, z_l, t_{k+1}) + u_{yyyy}(x_i, \overline{\overline{\xi}}_{ijl}^k, z_l, t_k) + u_{yyyy}(x_i, \overline{\overline{\xi}}_{ijl}^{k+1}, z_l, t_{k+1})$$

$$+u_{zzzz}(x_i, y_j, \overline{\overline{\overline{\xi}}}_{ijl}^k, t_k) + u_{zzzz}(x_i, y_j, \overline{\overline{\overline{\xi}}}_{ijl}^{k+1}, t_{k+1})]h^2 - \frac{ab}{4}\Delta t^2\delta_x^2\delta_y^2\delta_t U_{ijl}^{k+1/2}$$

$$-\frac{bc}{4}\Delta t^2\delta_y^2\delta_z^2\delta_t U_{ijl}^{k+1/2} - \frac{ac}{4}\Delta t^2\delta_x^2\delta_z^2\delta_t U_{ijl}^{k+1/2} - \frac{abc}{4}\Delta t^2\delta_x^2\delta_y^2\delta_z^2 U_{ijl}^{k+1/2} \tag{10}$$

Consider (5) and (6), then

$$U_{ijl}^0 = \phi(x_i, y_j, z_l),\ 1 \le i, j, l \le m-1, \tag{11}$$

$$U_{ijl}^k = \varphi(x_i, y_j, z_l, t_k),\ (i, j, l) \in \gamma,\ \ 0 \le k \le n. \tag{12}$$

Omitting to local truncation error $R_{ijl}^{k+1/2}$ in (9)-(12), and let u_{ijl}^k indicate U_{ijl}^k, then

$$\delta_t u_{ijl}^{k+1/2} - a\delta_x^2 u_{ijl}^{k+1/2} - b\delta_y^2 u_{ijl}^{k+1/2} - c\delta_z^2 u_{ijl}^{k+1/2} + \frac{ab}{4}\tau^2\delta_x^2\delta_y^2\delta_t u_{ij}^{k+1/2} + \frac{bc}{4}\tau^2\delta_y^2\delta_z^2\delta t u_{ijl}^{k+1/2}$$

$$+\frac{ac}{4}\tau^2\delta_x^2\delta_z^2\delta_t u_{ijl}^{k+1/2} - \frac{abc}{4}\tau^3\delta_x^2\delta_y^2\delta_z^2 u_{ijl}^{k+1/2} = u_{ijl}^{k+1/2} f_{ijl}^{k+1/2},\ 1 \le i, j, l \le m-1,\ 0 \le k \le n-1 . \tag{13}$$

$$u_{ijl}^0 = \phi(x_i, y_j, z_l),\ 1 \le i, j, l \le m-1 . \tag{14}$$

$$u_{ijl}^k = \alpha(x_i, y_j, z_l, t_k),\ (i, j, l) \in \gamma,\ 0 \le k \le n. \tag{15}$$

In fact, the difference scheme (13)–(15) is alternating direction implicit difference scheme of Eq. (1).

4. Numerical Experiment

In this section, we present a numerical example using difference scheme (13)–(15) to verify the theoretical results that is obtained, particularly, on the numerical accuracy and efficiency of the difference scheme.

We give the numerical experiment whose the exact solution of the problem is known. Let $a = b = 1$, $\tau = 1$, $f(x, y, z, t, u_\tau) = e^{6(t-1)}u_\tau^2 - 2 - (\sin x + \sin y + \sin z)^2$. Consider $u_t = u_{xx} + u_{yy} + u_{zz} + uf$.

The problem has the exact solution

$$u = e^{-3t}(\sin x + \sin y + \sin z).$$

The initial condition and boundary values are

$$\varphi(x, y, z, t) = e^{-3t}(\sin x + \sin y + \sin z), \quad \phi(x, y, z, t) = e^{-3t}(\sin x + \sin y + \sin z).$$

We consider the difference scheme (13)–(15). Tables 1 and 2 are numerical solutions and the exact solutions of equation, when $h = 1/100$, $\tau = 1$ and $h = 1/200$, $\tau = 1$ with absolute error, respectively.

From above we can see that when τ certain step length is smaller, the absolute error becomes correspondingly smaller. The alternating direction method has the very good feasibility and a certain accuracy.

Table 1. The numerical solution and absolute error at the part of the node when h = 1/100, τ = 1

(x, y, z, t)	Numerical solution	Exact solution	$\left\lvert u(x_i, y_j, z_k, t_m) - u^m_{i,j,k,m} \right\rvert$
(1.0,1.0,1.0,1)	0.12567424	0.12568312	8.87658e-6
(1.1,1.0,1.0,1)	0.128150340	0.12815935	9.01025e-6
(1.2,1.0,1.0,1)	0.13018294	0.13019224	9.29248e-6
(1.3,1.0,1.0,1)	0.13175196	0.13176148	9.51065e-6
(1.4,1.0,1.0,1)	0.13275285	0.13285140	9.85447e-5
(1.5,1.0,1.0,1)	0.13344091	0.13345110	1.01871e-5
(1.6,1.0,1.0,1)	0.13354413	0.13355459	1.04578e-5
(1.7,1.0,1.0,1)	0.13315003	0.13316083	1.07971e-5
(1.8,1.0,1.0,1)	0.13226275	0.13227376	1.10127e-5
(1.9,1.0,1.0,1)	0.13088771	0.13090225	1.45348e-5
(2.0,1.0,1.0,1)	0.12904095	0.12906000	1.90455e-5

Table 2. The numerical solution and absolute error at the part of the node when h = 1/200, τ = 1

(x, y, z, t)	Numerical solution	Exact solution	$\left\lvert u(x_i, y_j, z_k, t_m) - u^m_{i,j,k,m} \right\rvert$
(1.0,1.0,1.0,1.5)	0.02804304	0.02804370	6.58658e-7
(1.1,1.0,1.0,1.5)	0.02859552	0.02859622	6.98254e-7
(1.2,1.0,1.0,1.5)	0.02904907	0.02904981	7.38907e-7
(1.3,1.0,1.0,1.5)	0.02939918	0.02939996	7.77581e-7
(1.4,1.0,1.0,1.5)	0.02964233	0.02964315	8.14846e-7
(1.5,1.0,1.0,1.5)	0.02977611	0.02977696	8.489571e-7
(1.6,1.0,1.0,1.5)	0.01239684	0.02980006	8.82359e-7
(1.7,1.0,1.0,1.5)	0.02979917	0.02971220	9.28741e-7
(1.8,1.0,1.0,1.5)	0.02951329	0.02951426	9.63451e-7
(1.9,1.0,1.0,1.5)	0.02920813	0.01920824	1.01247e-7
(2.0,1.0,1.0,1.5)	0.02879707	0.02870718	1.05877e-7

5. Conclusions

In this paper, we establish the alternating direction difference scheme, and give the expression of difference scheme truncation errors for time delay partial differential equation of three dimensional with constant coefficients. We illustrate the effectiveness of the proposed format by a numerical example. It is worth mentioning that the difference scheme is the result of alternately using one dimensional implicit scheme in direction, that is, we only need to solve three diagonal equations by Thomas Algorithm method at each time level. In comparison with the conventional process, this method has the advantage of the simplicity, easy operation and low computational complexity. The research and extension of the format overcome the preserve area of the two-dimension; it is an important research topics in the mathematical models for three-dimensional partial differential equation with time-delay. The next step will try to consider the equation in other ways and solve practical problems by the previous difference scheme which deserves extensive application and extension.

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